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THE THEORY OF GROUND-WATER MOTION

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ABSTRACT

The existing analytical treatments of ground-water flow have mostly been founded upon the erroneous conception, borrowed from the theory of the flow of the ideal frictionless fluids of classical hydrodynamics, that ground-water motion is derivable from a velocity potential. This conception is in conformity with the principle of the conservation of matter but not with that of the conservation of energy. In the present paper it is shown that a more exceptionless analytical theory results if a potential whose value at a given point is defined to be equal to the work required to transform a unit mass of fluid from an arbitrary standard state to the state at the point in question is employed. Denoting this function by Φ, it is shown that the differential equation of fluid flow in an isotropic medium is given by $\mathbf{q} = -\sigma \operatorname{grad} \Phi$, where $\mathbf{q}$ is the flow vector whose magnitude is equal to the volume of fluid crossing a unit of area normal to the flow direction in unit time, and σ a specific conductivity parameter depending upon both the properties of the fluid and the medium. This is an expression of Darcy's law and is physically, as well as mathematically, analogous to Ohm's law in electricity and leads to the same deductions in analogous situations.

It is shown that $\sigma = k\rho/\eta$, where k is the permeability parameter depending upon the geometrical properties of the medium only and ρ and η are the density and viscosity, respectively, of the fluid.

The remainder of the paper is devoted to deducing the consequences of Darcy's law as just expressed, with particular regard for the practical problems of ground-water hydrology.

INTRODUCTION

Since the pioneer experiments of Darcy[1] on the flow of water through filter sands, a succession of important analytical treatments have appeared in which the flow of ground water has been discussed

[1] Henry Darcy, *Les Fontaines publiques de la ville de Dijon* (Paris: Victor Dalmont, 1856).

as a problem in field theory of mathematical physics. Outstanding among these analytical treatments is the now almost classical study of Slichter[2] and the recent treatments of Dachler[3] and of Muskat.[4]

While the importance of these and similar works cannot be overestimated, they have all fallen short in one or more important respects of the goal of establishing a theory of ground-water motion which is both free from internal contradictions and in conformity with all the fundamental principles of physics which ground-water motion must satisfy. In these treatments ample precaution has been taken not to violate the principle of the conservation of matter, but much less care has been exercised with respect to the equally inviolable first and second laws of thermodynamics.

In the present paper an attempt will be made to establish the theory of ground-water motion upon such a basis that the deduced consequences will be in more complete conformity with the principles of physics than has been the case heretofore. In doing this the entire subject will have to be re-examined from first principles because the contradictions to be avoided are inherent in the fundamental conceptions employed. The results we shall achieve will, in the main, resemble those of the existing treatments, but significant differences will arise where the neglect of energy relationships is nonallowable.

The present treatment will, as far as possible, be logically complete; yet the reader may occasionally feel the need for amplification of fundamental principles that can here be discussed only briefly. For this he is referred to standard treatises of potential theory, thermodynamics, hydrodynamics, dynamic meteorology, heat conduction, and electrodynamics. While many treatises of these various subjects exist, among those that the author has employed extensively are: Planck, *Theory of Heat* and *Thermodynamics;* Gibbs, "On the Equilibrium of Heterogeneous Substances" (in *Collected Works*); Kellogg, *Foundation of Potential Theory;* Coffin, *Vector Analysis;*

[2] C. S. Slichter, "Theoretical Investigation of the Motion of Ground Water," *19th Ann. Rept. U.S. Geol. Surv.*, Part II (1897–98), pp. 295–384.

[3] Robert Dachler, *Grundwasserströmung* (Vienna: J. Springer, 1936).

[4] Morris Muskat, *Flow of Homogeneous Fluids through Porous Media* (New York: McGraw-Hill, 1937).

Ewald, Pöschl, and Prandtl, *The Physics of Solids and Fluids;* Prandtl and Tietjens, *Hydro- and Aerodynamics* (2 vols.); V. Bjerknes, "Dynamic Meteorology and Hydrography" (*Carnegie Inst. Wash. Pub. 88* [1911]); Abraham and Becker, *Classical Electricity and Magnetism;* Page and Adams, *Principles of Electricity.*

For the geological background of ground-water hydrology no better source exists than the publications of the Division of Ground Water of the United States Geological Survey, particularly Meinzer, "The Occurrence of Ground Water in the United States with a Discussion of Principles" (*U.S. Geol. Surv. Water-Supply Paper 489* [1923]). A recent treatment of the same subject is Tolman, *Ground Water.*

THE DARCY EXPERIMENT

The general problem of ground-water motion is principally this: Suppose we are given the complete geometry (at least statistically) of an underground region extending from the earth's surface to some arbitrary depth, and we know the rate of the flow of water across the boundary of the region at all points, what is the nature of the flow at every point in its interior? A variation of this problem that is commonly encountered in practice consists in being given the geometry and the state of flow throughout a region in space and being required to anticipate the changes in the flow system which will result from certain specified alterations to be imposed upon it.

Stated thus broadly, both these problems appear formidable, and indeed are so until we develop certain necessary relationships which will enable us to deal with them. In order to acquire the necessary analytical tools for use in more general problems, we arbitrarily choose a simple flow system which we investigate exhaustively. Having done this, we then shall return with the knowledge thus gained to problems of a more complex nature.

The motion of ground water differs from the more familiar problems of fluid flow in that in this case the flow, instead of occurring in open basins or channels, takes place through an intricately branched network of open spaces interpenetrating a skeletal solid framework. There are, therefore, two distinct and essential elements to every such flow problem: the properties of the fluid, and the properties of the solid framework or *medium.* The flow properties of the fluid are sufficiently determined for most purposes by its viscosity, η, and its density, ρ.

The two more important properties of the medium are its *porosity* and its *permeability*. The porosity, ϵ, may be defined as the ratio of the free space, or voids, to the total volume. In rocks this ranges from zero to more than 50 per cent. The permeability of the medium pertains to the facility with which fluids flow through it. For the present only a qualitative definition is possible. Of two media through which the same fluid is made to flow under identical conditions, we may say that the one through which the flow is more rapid has the greater permeability. A medium may be said to be *isotropic* with respect to permeability if it is equally permeable to flow in all directions.

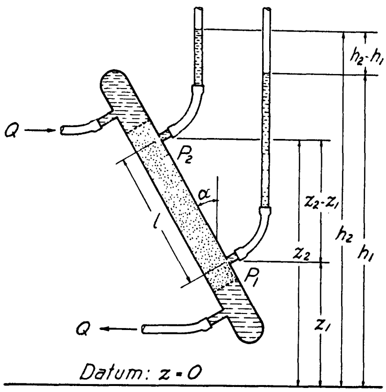

FIG. 1.—Apparatus for studying the flow of liquid through a permeable material.

Now, to begin our analysis, we choose a region of space occupied by a medium which is homogeneous and isotropic with respect to porosity and permeability. We assume, moreover, that the solid framework is insoluble and chemically inert with respect to the fluid passing through it, and ideally rigid. For the region to be investigated we choose a volume whose shape is that of a right circular cylinder of cross-sectional area A and length l, where both A and l are large, compared with the small irregularities or grain-size of the medium. We require that the flow through the volume be rectilinear and in a direction parallel to the axis of the cylinder.

These conditions may be approximately realized experimentally if we construct the apparatus shown in Figure 1. This consists of a large metal cylinder mounted so as to pivot about a horizontal axis perpendicular to its own axis. The mid-section of the cylinder between two screens is filled with sand or other suitable granular material of uniform grade. The ends of the cylinder are closed, but near each end inlet and outlet hose connections are attached. At points P_1 and P_2 in the same axial line and distance l apart two manometer openings are tapped into the mid-section filled with the permeable material. By means of flexible rubber tubing these are connected with permanently mounted open-topped manometers.

We choose a standard datum of elevation above which the elevations of the points P_1 and P_2 are z_1 and z_2, respectively. We let the angle of tilt of the large cylinder be α, where α is zero when the cylinder is vertical with P_2 above P_1.

Water is made to flow through the system at a total discharge rate of Q units of volume per unit of time. It rises to elevations h_1 and h_2 above the standard datum in the two manometer tubes terminated at P_1 and P_2, respectively.

We now wish to investigate the relations among the several variables of the system. We adopt the convention that the total discharge, Q, is positive when the flow is directed from P_1 to P_2, and negative when from P_2 to P_1. By varying Q we establish the following relationships: When

$$\left.\begin{array}{llll} Q = 0\,, & h_2 = h_1 & \text{or} & h_2 - h_1 = 0\,, \\ Q > 0\,, & h_2 < h_1 & \text{or} & h_2 - h_1 < 0\,, \\ Q < 0\,, & h_2 > h_1 & \text{or} & h_2 - h_1 > 0\,. \end{array}\right\} \tag{1}$$

We also observe that this relationship is one of proportionality and that

$$Q \propto -(h_2 - h_1)\,. \tag{2}$$

We next investigate the effect of changing the angle of tilt α. Letting Q remain constant, we tilt the system so that α assumes all possible values from zero to 180°, and we discover that the results expressed by equations (1) and (2) are in nowise influenced by the value of α, being exactly the same whether the system is upright, inclined, horizontal, or upside down.

Two other important variables of the system whose effects require investigation are the area of cross section A and the length l. This could be done by rebuilding the apparatus and varying first A and then l; but to do so is hardly necessary, since the answers are implicit in the experiment already described.

Consider first the variation of A. Since the direction of the flow is axial, no change will be produced if we insert a thin axial partition dividing the cylinder into two halves of area $A/2$ each and discharge $Q/2$ in each half. In this case the manometer readings, which remain unchanged, are registering the flow in only one-half of the system. Still further partitioning would give us a series of n equal parallel tubes, each conducting water at a rate Q/n, with still no change in the manometer readings, although only the flow Q/n in a tube whose cross section is A/n is being recorded. From this it is clear that a reduction in cross section of the flow system has no influence on the manometer readings provided the discharge across that area is varied at the same rate. This leads us to the conclusion that it is the ratio Q/A which must be kept constant if the manometer readings are not to vary. The total discharge divided by the cross-sectional area is obviously equal to the discharge per unit of area or to the *specific discharge* q. Consequently,

$$\frac{Q}{A} = q \propto -(h_2 - h_1) \tag{3}$$

or, for a given value of the manometer readings,

$$Q = qA \, , \tag{4}$$

where the specific discharge, q, is the constant of proportionality.

In a similar manner we investigate the effect of varying the length l between the bases of the two manometer tubes. The height difference $h_2 - h_1$ is the reaction to the discharge q through the length l. Since the flow is uniform over every cross section of the system, then for length dl there must correspond a fall in h, of dh, such that

$$\frac{dh}{dl} = \frac{h_2 - h_1}{l} \, , \tag{5}$$

and these quantities must be proportional to q. Or, if $(h_2 - h_1)$ is kept constant,

$$Q \propto \frac{1}{l} \, . \tag{6}$$

Now, if we combine the results of equations (2)–(6) into a single expression, we obtain

$$Q = -KA \cdot \frac{h_2 - h_1}{l}; \tag{7}$$

or, by employing the specific discharge q and the differential expression dh/dl,

$$q = \frac{Q}{A} = -K \cdot \frac{dh}{dl}, \tag{8}$$

where K is a constant of proportionality.

With the exception of the variation of the angle of tilt, α, which, as we have seen, produces no effect, the experiment just described is essentially that performed originally by Henry Darcy, whose results are better expressed in his own words:

Ainsi, en appelant e l'eppaiseur de la couche de sable, P la pression atmosphérique, h la hauteur de l'eau sur cette couche, on aura $P + h$ pour la pression a laquelle sera soumise la base superieure; saient, de plus, $P \pm h_0$ la pression supportié par la surface inferieure, k un coefficient dependant de la perméabilité de la couche, q le volume débité, on a

$$q = k\frac{s}{e}[h + e \pm h_0]$$

qui se réduit à

$$q = k\frac{s}{e}[h + e], \quad \text{quand} \qquad h_0 = 0,$$

au lorsque la pression sous le filtre est equal à la pression atmosphérique.[5]

Darcy explains elsewhere that s is the area of cross section. Although the language employed is necessarily somewhat archaic by present standards, it seems clear that his standard datum from which the manometer heights were measured was the lower base of his filter sand. Then $h + e$ is the height in the upper manometer, and h_0 that in the lower, this being either positive or negative, depending upon whether or not a vacuum was employed. It is clear, therefore, that the results announced by Darcy are entirely in accord with those expressed by equations (7) and (8)—a relationship which appropriately has come to be known as "Darcy's law."

[5] *Op. cit.*, pp. 570 ff.

Darcy also noted, significantly, that the relationship was no longer valid for fluid velocities greater than 10–11 cm/sec.

THE PHYSICAL SIGNIFICANCE OF DARCY'S LAW

The relationships embodied in equation (8) form the necessary basis upon which any analytical theory of ground-water motion must rest; but the method of obtaining this equation has been empirical, and, expressed in this primitive form, the equation is of slight usefulness because it expresses only what we have learned already and gives us no insight into the deeper mechanism of fluid flow.

What determines the direction of the fluid flow in the first place? What would be the effect if we changed from a finer to a coarser sand? How would the flow rate be altered if we changed the viscosity or the density of the fluid? These and other similar ones are questions that equation (8) does not answer; yet they are questions to which it is highly important that answers be known.

Let us consider first the question of what determines the direction of the fluid motion, that is, whether the flow in Figure 1 is to be directed from P_1 to P_2 or from P_2 to P_1. We obviously cannot say that elevation is the determining factor, because, as we have seen already, if the flow is initially from a higher to a lower elevation, an inversion of the system changes the flow from a lower to a higher elevation without affecting in the slightest degree either its rate or the readings of the manometers.

If it is not elevation, perhaps it is the fluid pressure that is the determining factor in the flow, with the flow always directed away from regions where the pressure is higher and toward those where it is lower. In fact, the great majority of all writers upon this subject have stated that this is so, and many have employed equations of the form

$$q = -K' \cdot \frac{dp}{dl} \tag{9}$$

as a statement of Darcy's law, presumably under the impression that equations (8) and (9) are physically equivalent statements.

Whether or not this is so may readily be determined in the following manner: At any point P in a flow system whose elevation above the standard datum is z, we terminate a manometer tube in which the liquid rises to a height h above the datum. The pressure at the point P is determined by the height of the liquid column above P and is given by

$$p = \rho g(h - z) + p_0 \,, \tag{10}$$

where p is the pressure, ρ the density of the liquid, g the acceleration due to gravity, and p_0 the pressure of the atmosphere. Now, if we apply equation (10) to the points P_1 and P_2 of Figure 1, we obtain

$$\left.\begin{aligned} p_1 &= \rho g(h_1 - z_1) + p_0 \,, \\ p_2 &= \rho g(h_2 - z_2) + p_0 \,, \end{aligned}\right\} \tag{11}$$

and the difference between the pressures at the two points is

$$p_2 - p_1 = \rho g(h_2 - h_1) - \rho g(z_2 - z_1) \,. \tag{12}$$

Now we already know that as $(h_2 - h_1) \to 0$, $q \to 0$; but in that case p_1 is less than p_2 by the quantity $\rho g(z_2 - z_1)$, which gives us a case where the flow is zero when one of the two pressures is greater than the other. Next, suppose we make h_2 slightly greater than h_1, which will correspond to flow from P_2 to P_1, but still not great enough for $(h_2 - h_1)$ to equal $(z_2 - z_1)$. In this case the pressure p_2 will still be less than p_1, and we shall have flow from a region of lower to one of higher pressure. Then if we invert the system by rotation through 180° about a horizontal axis, the pressure at P_2 will become greater than that at P_1 without the rate of flow or the manometer reading being changed. Consequently, we must conclude that a liquid can with equal facility be made to flow from a region of higher to one of lower pressure, or from a region of lower to one of higher, quite arbitrarily.

To investigate equation (9) we need only to express equation (12) differentially:

$$\frac{dp}{dl} = \rho g \cdot \frac{dh}{dl} - \rho g \cdot \frac{dz}{dl} \,. \tag{13}$$

But

$$\frac{dz}{dl} = \cos \alpha , \tag{14}$$

so that when we insert this into equation (13) we obtain

$$\frac{dp}{dl} = \rho g \cdot \frac{dh}{dl} - \rho g \cos \alpha . \tag{15}$$

When this is substituted into equation (9), it gives

$$q = -K' \rho g \left(\frac{dh}{dl} - \cos \alpha \right) , \tag{16}$$

which is manifestly not equivalent to equation (8) under any condition except when the second term, that of $\cos \alpha$, is negligible, compared with the first. This is only true either when $\cos \alpha$ is approximately zero, corresponding to nearly horizontal flow, or when dh/dl is very large, compared with unity, the maximum value of $\cos \alpha$.

In the experiment of Figure 1, and in ground-water motion in general, the direction of flow may have any inclination from vertical to horizontal, so that the assumption of $\cos \alpha = 0$ is not tenable. Moreover, dh/dl is ordinarily very much smaller than unity, the maximum value of $\cos \alpha$. Hence, in problems of ground-water flow, equation (9) is not only not equivalent to Darcy's law, as expressed by equation (8), but is not even a valid approximation to it. Consequently, the employment of equation (9) or any equivalent statement wherein the rate of flow is assumed to be proportional to the pressure gradient, when dealing with ground-water problems, is to be ruled out on the grounds of being physically erroneous.

When dealing with the flow of gases, on the other hand, dh/dl, or its equivalent, may be large, compared with $\cos \alpha$, in which case the employment of equation (9) becomes an allowable approximation.

What, then, does determine the direction of the flow? What we still seek to find is some physical quantity, capable of measurement at every point in a flow system, whose properties are such that the flow always occurs from regions in which the quantity has higher values to those in which it has lower, regardless of the direction in space. What we have demonstrated so far is that neither elevation nor pressure is such a quantity.

Formally, the manometer height h satisfies this condition entirely, but to adopt it empirically without further investigation would be like reading the length of the mercury column of a thermometer without knowing that temperature was the physical quantity being indicated. The quantity we seek, therefore, is evidently the hidden physical quantity whose magnitude is indicated by the height h of the liquid in the manometer tube, measured from a standard datum.

The clue to this quantity is to be found if we direct our attention momentarily to an apparently irrelevant experiment. Suppose we take a simple pendulum, start it swinging with a wide amplitude, and then leave it entirely free from outside disturbance. We record its motion by means of time exposures taken periodically, the time of each exposure being made equal to the period of the pendulum. We shuffle the photographs and ask a second person to arrange them in chronological order. This is illustrated by Figure 2, where A, B, C, D, and E are prints selected at random, and the problem is to determine the chronological order in which the exposures were made.

By inspection we would say that the order was D, B, E, A, and C, because this is the order of decreasing amplitude of swing, and our experience tells us that an isolated pendulum always swings with decreasing amplitude—never the reverse.

We have, therefore, an experimental process which only goes in a single direction, and it is this unidirectional characteristic that we wish to consider. This we may best do by taking into account the energy transformations of the process. To set the pendulum swinging in the first place, we had to give it an initial supply of potential energy. As it swung back and forth, there were periodical transformations of potential energy to kinetic energy and back to potential energy again, the sum of the two comprising the mechanical energy of the system and remaining constant for constant amplitude of swing. But we have noted already that the amplitude continuously diminished; consequently, there must have been a continuous dissipation of the mechanical energy initially supplied to the system. Also, ultimate equilibrium corresponds to a state of rest with the pendulum at its lowest possible position, that is to say, with the kinetic energy of the system equal to zero, and the potential energy the minimum possible compatible with the constraints of the system.

We know, moreover, that the progressive decrease in the amplitude of swing is due to frictional resistances; and, by the principle of the conservation of energy, we know that the mechanical energy lost by the system reappears as heat at the temperature of the surroundings. Now, if we could transform this thermal energy back to mechanical energy again without other permanent changes, then we could remotivate the pendulum, allowing it to use over and over again the same energy, which would be a form of perpetual motion.

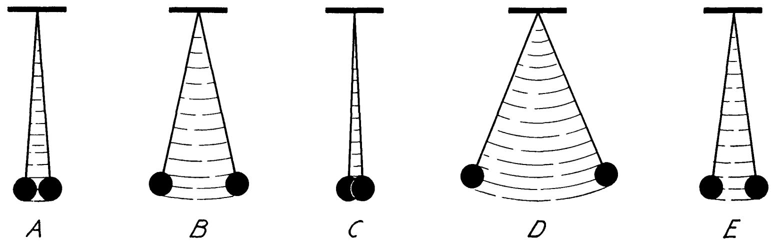

FIG. 2.—Unidirectional and irreversible transformation of isolated mechanical system.

Our purely negative experience in this connection leads to the conclusion that such a process is impossible, and a generalization of this fact so as to apply to all manner of processes gives us the Second Law of Thermodynamics, which states, in essence, that any material transformation which involves friction, or its equivalent, is unidirectional and irreversible in character, meaning that, once the process has taken place, by no method whatsoever can it be undone again. In other words, if we have any initial configuration of matter in an isolated system, and this configuration undergoes spontaneous change accompanied by friction, it is quite impossible ever to restore the matter contained within that system to its initial configuration and at the same time leave the material configuration external to that system undisturbed.

It is this method of reasoning that we now wish to employ with respect to our problem of fluid flow. The flow of a fluid is a mechanical process; and, when the fluid has viscosity and the flow occurs through the small passages of porous rocks, friction is not only present but is one of the dominant influences in the process. Consequently, if such flow occurs at all, it must be accompanied by an irreversible transformation of mechanical to thermal energy through the mechanism of fluid friction. Furthermore, like the pendulum, final equilibrium must correspond to a configuration in which the kinetic energy is zero; and the potential energy, the minimum possible compatible with the constraints of the system. Hence, to have flow, there must have been supplied an initial store of mechanical energy, and the changes that occur thereafter must always be in the direction of a decrease by dissipation into heat of this mechanical energy.

So far this gives the generalized direction of the process but not the geometrical direction of the flow in space. The latter is obtained, however, when we consider that the mechanical energy of the system is associated with the elements of its mass or volume and that these occupy particular positions in space. The direction of the flow in space must therefore be away from regions in which the mechanical energy per unit of mass is higher and toward regions in which it is lower.

It is not to be inferred that the energy which a system possesses in consequence of the fact that an element of mass occupies a given position resides upon or within the mass element. For example, the potential energy $U = gz$ of a unit mass due to the earth's gravitational field is an energy of the unit mass and the earth considered as a single system; yet we may unambiguously refer this energy to the unit mass at a specified position. This same is true of the fluid system with which we now deal.

The mechanical energy of the fluid per unit of mass is, therefore, evidently the physical quantity we set out to find; and hereafter we shall refer to it as the *potential* of the fluid in question, there being as many distinct potentials as there are distinct fluids to be dealt with.

Our problem now resolves itself into the determination of the fluid potential, or mechanical energy per unit of mass, of the given fluid at any arbitrary point in space. To obtain this we note first that energy is a relative quantity and is measurable by the amount of work required to effect any given transformation from some arbitrary initial state to a specified final state. The potential of a fluid at a specified point is, accordingly, the work required to transform a unit of mass of the fluid from an arbitrarily chosen standard state to the state at the point under consideration. For the standard state it is convenient to employ an elevation of zero, a pressure of 1 atmosphere, and a velocity of zero (relative to the earth's surface). Let the fluid in its final state at the point P be characterized by an elevation z, a pressure p, and a velocity v. We may also let V_0 be the volume per unit mass, or specific volume, of the fluid in its standard state, and V be that for the final state. Let the corresponding densities be ρ_0 and ρ. We may also note that the density is the reciprocal of the specific volume: $V = 1/\rho$.

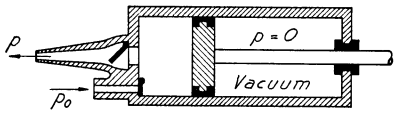

FIG. 3.—Pump for transforming liquid from standard to final state.

We wish to find the work required to transform a unit mass of the fluid from the initial to the final state, and to do this we imagine a pump constructed along the lines indicated by Figure 3. This consists of a cylinder with frictionless piston, on the front of which is the fluid chamber and on the back a perfect vacuum. Inlet and outlet valves are provided. We then imagine the transformation to be effected by the following successive steps:

1. Under standard conditions we slowly withdraw the piston and charge the cylinder with unit mass of the fluid. The work done by the piston on the fluid is then

$$w_1 = -p_0 V_0 . \tag{17}$$

2. Next we lift the pump with its fluid contents to the point P of elevation z. The work expended for this is

$$w_2 = +gz + m_p gz , \tag{18}$$

where gz is the work required to lift the unit mass of the fluid, and $m_p gz$ that required to lift the pump alone.

3. The contents of the cylinder are injected into the system at point P. The work required for this is

$$w_3 = + \int_V^{V_0} p \cdot dV + pV \,. \tag{19}$$

The first term to the right of equation (19) is the work of compression of the fluid in order to raise its pressure from p_0 to p before it can be injected. The pV term is the work of injection against the pressure p.

4. The fluid is accelerated from a velocity of zero to that of v, requiring an amount of work

$$w_4 = + \frac{v^2}{2} \,. \tag{20}$$

5. The cylinder is returned to its initial position at zero elevation, thus completing the cycle. This requires an amount of work

$$w_5 = -m_p g z \,. \tag{21}$$

The sum of these separate amounts of work is the potential Φ of the fluid at the point P. Performing the addition and canceling out terms that repeat with opposite signs gives us

$$\Phi = gz - p_0 V_0 + \int_V^{V_0} p dV + pV + \frac{v^2}{2} \,. \tag{22}$$

In this, the first and last terms on the right-hand side are the gravitational potential energy and the kinetic energy, respectively. The significance of the other three terms is best visualized by means of the "indicator diagram" of Figure 4, in which the pressure in the cylinder is plotted against piston displacement for both compressible and incompressible fluids. If the fluid is incompressible, a condition satisfied approximately by liquids under ordinary pressure ranges,

$$\int p \cdot dV = 0 \qquad \text{and} \qquad V = V_0 \,. \tag{23}$$

In this case the pressure-volume work reduces to $(p - p_0)V$, and equation (22) simplifies to

$$\Phi = gz + (p - p_0)V + \frac{v^2}{2} . \tag{24}$$

By a mathematical transformation we can convert equation (22) into another form whose physical significance may not be im-

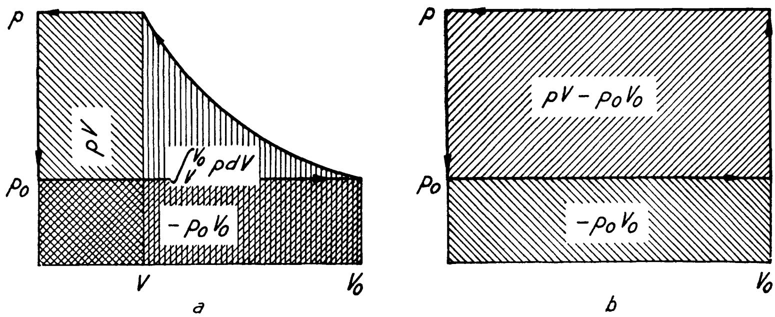

FIG. 4.—Pump indicator diagrams: *a*, for gases; *b*, for liquids

mediately apparent but which will prove to be of great usefulness later. To effect this we make use of the fact that

$$d(pV) = p \cdot dV + V \cdot dp ,$$

or

$$\int p \cdot dV = \int d(pV) - \int V \cdot dp . \tag{25}$$

Then, as a definite integral, this becomes

$$\left.\begin{aligned} \int_V^{V_0} p \cdot dV = \int_{pV}^{p_0V_0} d(pV) - \int_p^{p_0} V \cdot dp = p_0V_0 - pV \\ + \int_{p_0}^{p} V \cdot dp . \end{aligned}\right\} \tag{26}$$

Substituting this value for $\int_V^{V_0} pdV$ into equation (22) transforms that equation into

$$\Phi = gz + \int_{p_0}^{p} V \cdot dp + \frac{v^2}{2} , \tag{27}$$

which, when we substitute $1/\rho$ for V, becomes

$$\Phi = gz + \int_{p_0}^{p} \frac{dp}{\rho} + \frac{v^2}{2}. \qquad (28)$$

A graphical interpretation of equation (26) is readily afforded by noting that the area enclosed by the indicator diagram of Figure 4, *a*, is equal to the work performed by the pump per cycle and that this is given by $\int_{p_0}^{p} V \cdot dp$ or by $\int_{p_0}^{p} \frac{dp}{\rho}$.

Equation (28) is an expression for the fluid potential at a point P in the most general form that we shall require. In order for the value of Φ so defined to be unique, however, it is necessary to stipulate that the density of the fluid must be a function of the pressure only, for otherwise the value of $\int_{p_0}^{p} V dp$ will be indeterminate. For liquids this condition is automatically complied with to the extent that their densities may be regarded as constant. For gases it is satisfied by isothermal conditions, which for many problems is a satisfactory approximation, and also by adiabatic conditions, which are of great importance in meteorology.

If the fluid flows without friction, as is approximately the case for liquids of small viscosity and gases, when moving in large open spaces, then each element of mass retains its mechanical energy undiminished, and along any given path of fluid flow each unit of mass will have the same energy and potential. For that case

$$\Phi = gz + \int_{p_0}^{p} \frac{dp}{\rho} + \frac{v^2}{2} = \text{constant}, \qquad (29)$$

which is a generalization of a celebrated theorem announced in 1738 by Daniel Bernoulli relating the elevation, pressure, and velocity along a given flowline of a fluid in frictionless flow.

If the flow is not frictionless, as is pre-eminently the case with ground water and similar fluid flow, then the mechanical energy initially possessed by an element of the fluid must continuously be dissipated as this element traverses its path of flow. Consequently, the value of Φ in this instance must continuously decrease in the direction of the flow.

Another consequence of friction in ground-water motion is to damp out any large velocities, the flow velocity of ground water being rarely as great as 1 cm/sec and commonly of the order of a few centimeters per day. For our purposes we can accordingly neglect the kinetic energy term $v^2/2$ as being negligible in comparison with the other terms of the equation for the potential. This simplifies the expression for the potential down to

$$\Phi = gz + \int_{p_0}^{p} \frac{dp}{\rho}, \tag{30}$$

which, for liquids, reduces still farther to

$$\Phi = gz + \frac{p - p_0}{\rho}. \tag{31}$$

Now, in order to be able to measure Φ at any point P of elevation z and pressure p in a liquid flow system, suppose we return to our earlier device of terminating at the point P a manometer tube into which the liquid rises to the height h above the standard datum. The pressure p in terms of the manometer reading and the elevation z is then given by equation (10). When this is substituted into equation (31), we obtain the remarkably simple result

$$\Phi = gz + \frac{[\rho g(h - z) + p_0] - p_0}{\rho} = gh. \tag{32}$$

Hence the fluid potential is indeed the hidden physical quantity we originally set out to discover, since its magnitude is indicated by the height h of the manometer and is numerically equal to h multiplied by the acceleration due to gravity.

With this background, let us return now to the results of the Darcy experiment as expressed empirically by equation (8). From equation (32)

$$\frac{dh}{dl} = \frac{1}{g} \cdot \frac{d\Phi}{dl}, \tag{33}$$

which, when employed in conjunction with equation (8), gives the two equivalent alternative expressions of Darcy's law:

$$q = -K \cdot \frac{dh}{dl} = -K \cdot \frac{1}{g} \cdot \frac{d\Phi}{dl} . \tag{34}$$

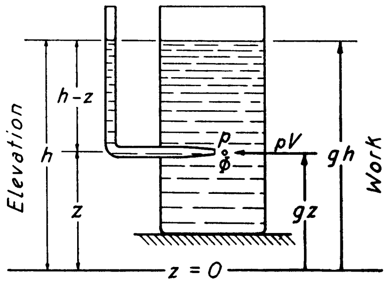

FIG. 5.—Fluid potential at any point inside a body of static liquid

ANALYSIS OF THE PARAMETER K

Our next task is to analyze the parameter K. In the experiment of Figure 1 we investigated certain variables whose effects are stated explicitly in equation (8), but in performing this experiment we arbitrarily kept all other factors constant except the particular ones under investigation. The experiment, however, was incomplete in that a number of possible variables were not taken into consideration. What, for example, would have been the effect of a change of the density or the viscosity of the fluid, of the coarseness of the sand, or of other possible variables of the system?

Manifestly, all these other variables were included in the stipulation "all else remaining constant"; and their combined effects are lumped together into the "constant" of proportionality K, which is accordingly, therefore, not a true constant but a variable parameter whose value depends upon those of the lumped variables. It is important that we find these hidden variables so that we can include them explicitly in our flow equation, enabling us to eliminate K entirely. To do this we must delve somewhat more deeply than heretofore into the mechanics of fluid flow.

In what follows, two different points of view—the microscopic and the macroscopic—distinguished from one another by the size scale adopted, will prove useful. When we employ the microscopic point of view, we shall be concerned with fluid elements that are large compared with the kinetic irregularities of molecular motion but smaller than the open passages of the medium. When we employ the macroscopic point of view, the fluid elements we speak of shall be large enough that the irregularities of flow due to the medium need not be considered, but only the statistical resultant.

In order to determine what the various factors are that influence the rate of flow, and the part they play, we need to know what the forces are that act upon the fluid elements and upon what these depend. To determine these we shall stipulate that the flow through the sand in the apparatus of Figure 1 be kept *steady*, that is, not changing with time, which will be accomplished if we maintain q constant at any arbitrarily chosen value. Then we adopt the microscopic point of view in order to examine the flow in some detail.

When seen on this scale, the flow will consist of the passage of the fluid through an intricate, branching, three-dimensional network of interstices, analogous to the two-dimensional flow of a river through a complex of small islands. If we choose a particular point, fixed with respect to the framework, then a fluid particle passing through this point will follow a definite path, or *streamline*, and every other particle passing through the same point will transverse the same path and assume the same velocities at corresponding points along the path. Consequently, to every fixed point within the flow system there corresponds a particular fluid velocity, both in magnitude and direction, and through each point there passes a particular streamline. We may accordingly think of the flow as being represented by a family of streamlines, one passing through each point, or by a *field of velocity vectors*, one terminated at each fixed point occupied by the fluid.

In addition to this, around any fixed point we may take a small volume element, also fixed with respect to the framework. While the fluid continuously flows through such an element, at every instant the volume element is occupied by a particular element of the fluid. The forces acting upon this element are of two kinds: those that act upon its surface, or *surface* forces, and those that act upon the mass within the volume element, or *body* forces. If dV is the volume of the element, then $\rho \cdot dV$ will be its mass, and by Newton's laws of motion the sum of the applied forces is equal to the product of the mass by its acceleration. Then

$$\rho \cdot dV \cdot \mathbf{a} = \mathbf{f}_a + \mathbf{f}_b \,, \tag{35}$$

where the vector $\mathbf{a}$ is the acceleration and the vectors $\mathbf{f}_a$ and $\mathbf{f}_b$ the surface and body forces, respectively.

Now, by the D'Alembert principle, we can introduce a fictitious force

$$\mathbf{f}_i = -\rho \cdot dV\mathbf{a} \,, \tag{36}$$

which is the reaction due to inertia of the body to the applied forces $\mathbf{f}_a$ and $\mathbf{f}_b$. Then we have

$$-\mathbf{f}_i = \mathbf{f}_a + \mathbf{f}_b \,, \qquad \text{or} \qquad \mathbf{f}_i + \mathbf{f}_a + \mathbf{f}_b = 0 \,, \tag{37}$$

which tells us that the moving element of fluid is in a state of dynamic equilibrium under these three forces in a manner quite analogous to the more familiar static equilibrium.

A more useful classification of the forces acting upon the fluid element is upon the basis of function. On this basis we may speak of a *driving* force $\mathbf{f}_d$, a resistive force arising from frictional resistance due to the fluid viscosity, $\mathbf{f}_r$, and a reactive force due to the inertial reaction to acceleration $\mathbf{f}_i$. We may think of $\mathbf{f}_d$ as being the independently variable force which supplies the energy to the fluid element. Both $\mathbf{f}_r$ and $\mathbf{f}_i$ are dependent variables, depending only upon the state of motion of the fluid and being zero when the fluid motion is zero. The inertial force, $\mathbf{f}_i$, neither contributes energy to nor subtracts it from the system, whereas $\mathbf{f}_r$ is the energy-dissipating force.

From this it is clear that $\mathbf{f}_d$ and $\mathbf{f}_r$ are the principal forces from the point of view of function, while $\mathbf{f}_i$ plays the role of a superimposed modifying influence.

In order to discover the factors which determine $\mathbf{f}_i$ and $\mathbf{f}_r$, the microscopic viewpoint is necessary. The driving force, $\mathbf{f}_d$, can also be deduced from this viewpoint, but to do so is unduly complicated.

We shall accordingly investigate the forces $\mathbf{f}_i$ and $\mathbf{f}_r$ acting upon a microscopic element and shall then extend our results by integration to a macroscopic volume element before deriving $\mathbf{f}_d$. On a microscopic scale, however, when $\mathbf{f}_i$ and $\mathbf{f}_r$ are known, $\mathbf{f}_d$, being the sum of these two, is uniquely determined.

In order to investigate the forces due to inertia and to viscosity, let us erect co-ordinate axes with the x-axis parallel to the direction

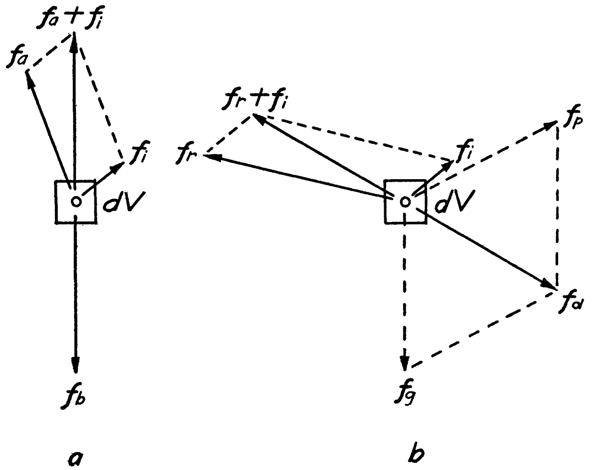

FIG. 6.—The forces which act upon a microscopic fluid element classified a as surface force, body force, and inertial force and b as driving force, resistive force, and inertial force.

of fluid flow. We let the volume element dV have lengths of side dx, dy, and dz. We let u, v, and w be the components of the velocity parallel to the x-, the y-, and the z-axes, respectively.

Taking first the force due to inertia, we resolve the force and the acceleration upon which it depends into components parallel to the separate axes. Taking the x-component,

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial t} , \qquad (38)$$

where $\partial u/\partial t$ signifies the change of u with time at a particular point, $\partial u/\partial x$ the change of u with x at a particular time, and $\partial x/\partial t$ the change of the x-co-ordinate with time, of a particular particle, and so for the other terms. But

$$\left.\begin{aligned} \frac{\partial x}{\partial t} &= u\,, \\ \frac{\partial y}{\partial t} &= v = 0\,, \\ \frac{\partial z}{\partial t} &= w = 0\,, \end{aligned}\right\} \tag{39}$$

and, for steady motion,

$$\frac{\partial u}{\partial t} = 0\,. \tag{40}$$

Then, in this special case equation (38) simplifies to

$$a_x = u \cdot \frac{\partial u}{\partial x}\,, \tag{41}$$

and the force component is

$$f_{ix} = -\rho \cdot dV \cdot u \cdot \frac{\partial u}{\partial x}\,. \tag{42}$$

Similar expressions obtain for the components parallel to each of the other axes, but all that concerns us is the *form* of equation (42), so that we do not need to consider the matter in greater detail.

Now let us consider the force due to viscosity. Upon the y and z faces of the volume-element shear stresses will act. Let τ_{yx} be the shear stress parallel to the x-axis upon the y-face whose outward directed normal is toward the negative end of the y-axis. Upon the opposite face let the corresponding stress be $\tau_{yx} + (\partial\tau_{yx}/\partial y)dy$. Then the net force parallel to the x-axis produced by these two stresses will be

$$f_{ryx} = \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} \cdot dy\right) dxdz - \tau_{yx} \cdot dxdz = +\frac{\partial \tau_{yx}}{\partial y} \cdot dV\,. \tag{43}$$

Complete elaboration would provide additional terms, but again they would all be of the form of the right-hand term of equation (43).

Our next problem is to relate the shear stress acting upon the surface of a fluid element with the state of its flow. By Newton's law of viscosity the rate of shear in a viscous fluid is proportional to the intensity of the shear stress acting upon it. If γ_{yx} is the change of

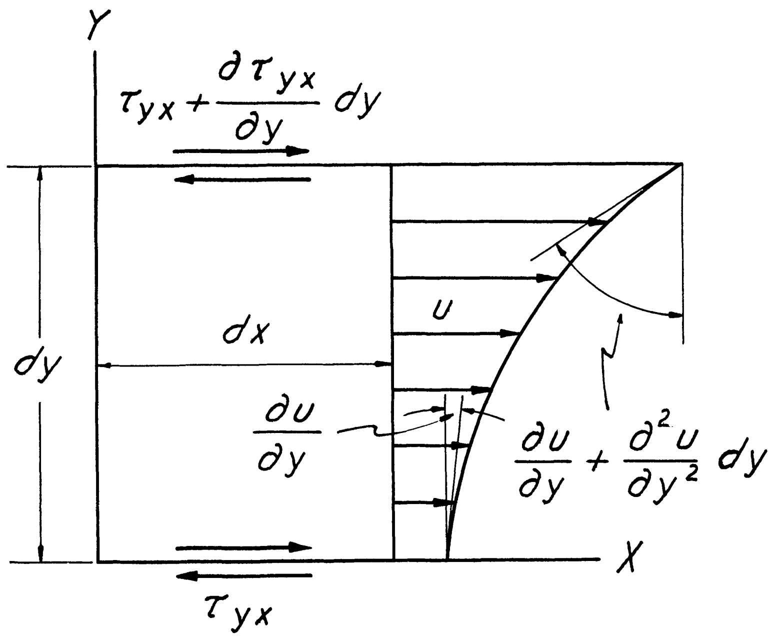

FIG. 7.—Manner of dependence upon the fluid velocity gradients of shearing stresses and unbalanced frictional forces acting upon microscopic fluid element.

an original right angle in the fluid with sides parallel to the x- and y-axes, then $d\gamma_{yx}/dt$ is the time rate of the shear, and

$$\tau_{yx} = \eta \cdot \frac{d\gamma_{yx}}{dt}, \tag{44}$$

where the constant of proportionality η is the viscosity of the fluid. But

$$d\gamma_{yx} = \frac{\left(u + \frac{\partial u}{\partial y} \cdot dy\right) dt - udt}{dy} = \frac{\partial u}{\partial y} \cdot dt, \tag{45}$$

from which we obtain

$$\frac{d\gamma_{yx}}{dt} = \frac{\partial u}{\partial y}, \tag{46}$$

which, when substituted into equation (44), gives

$$\tau_{yx} = \eta \cdot \frac{\partial u}{\partial y}. \tag{47}$$

Now if we differentiate this with respect to y,

$$\frac{\partial \tau_{yx}}{\partial y} = \eta \cdot \frac{\partial^2 u}{\partial y^2}, \tag{48}$$

which can then be substituted into equation (43), with the result that

$$f_{ryx} = \eta \cdot \frac{\partial^2 u}{\partial y^2} \cdot dV. \tag{49}$$

What we are interested in finding is the manner in which the inertial and resistive forces vary with the specific discharge q and with a change of scale of the medium. For generality, suppose that we have two *geometrically similar* media of different-length scales, and through these we cause two different fluids to undergo *kinematically similar* flow. Geometrical similarity requires that all corresponding lengths of the two systems must have a constant ratio

$$\frac{l_2}{l_1} = \frac{d_2}{d_1}, \tag{50}$$

where d_1 and d_2 are corresponding grain diameters of the two systems, these being taken as characteristic lengths.

Kinematic similarity requires the two flow systems to be exact replicas of one another except for the length and time scales. All streamlines must be geometrically similar, and all corresponding velocities must have identical directions and proportional magnitudes. Since the specific discharge q has the dimensions

$$[q] = \left[\frac{\frac{\text{Volume}}{\text{Area}}}{\text{Time}}\right] = [LT^{-1}] = [\text{Velocity}], \tag{51}$$

this represents a generalized velocity. Consequently, in kinematically similar flow all corresponding velocities must bear to each other the ratio

$$\frac{v_2}{v_1} = \frac{q_2}{q_1}. \tag{52}$$

Finally, since forces act at all points upon a fluid in motion, in order for two flows to be kinematically similar, it is necessary for all corresponding forces of the two systems to be similar, that is, to have identical directions and proportional magnitudes:

$$\frac{f_{i2}}{f_{i1}} = \frac{f_{r2}}{f_{r1}} = \frac{f_{d2}}{f_{d1}}. \tag{53}$$

This also necessitates that the corresponding force-equilibrium triangles of the two systems must be similar, so that

$$f_{i1} : f_{r1} : f_{d1} :: f_{i2} : f_{r2} : f_{d2} \tag{54}$$

or

$$\frac{f_{i1}}{f_{r1}} = \frac{f_{i2}}{f_{r2}}. \tag{55}$$

The manner in which the terms of equation (55) depend upon the velocity and the length scales is given by equations (42) and (49). For either of the two flow systems

$$\frac{f_i}{f_r} \propto \frac{f_{ix}}{f_{ryx}} = \frac{\rho u \cdot \dfrac{\partial u}{\partial x} \cdot dV}{\eta \dfrac{\partial^2 u}{\partial y^2} \cdot dV}. \tag{56}$$

In this

$$u \propto q, \qquad \text{and} \qquad \frac{\partial u}{\partial x} \propto \frac{q}{d}. \tag{57}$$

Similarly,

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) \propto \frac{q}{d^2}. \tag{58}$$

Substituting these results into equation (56) gives

$$\frac{f_i}{f_r} \propto \frac{\rho \cdot \frac{q^2}{d}}{\eta \cdot \frac{q}{d^2}} = \frac{qd}{\frac{\eta}{\rho}} = R\,. \tag{59}$$

Since equations (56)–(59) are identical for the two kinematically similar systems, then it follows that the quantity R defined by equation (59) must be the same for each. R is known as the Reynolds' number of the system, so called in honor of Osborne Reynolds,[6] whose pioneer studies first disclosed its significance.

It is important that the generality of equation (59) be appreciated, for, since it applies equally well to either of two kinematically similar systems, then

$$\frac{q_1 d_1}{\frac{\eta_1}{\rho_1}} = \frac{q_2 d_2}{\frac{\eta_2}{\rho_2}} = R\,, \tag{59a}$$

even though the length scales, the discharges, and the viscosity and density of the fluids may differ widely in the two cases.

Conversely, any two states of flow through geometrically similar frameworks will be kinematically similar when both have the same value of Reynolds' number; but for the same fluid in the same framework, since R is proportional to q, this condition cannot strictly be satisfied for different velocities of flow. Practically, it is satisfied for very small values of R, since, while both f_i and f_r tend to zero as q tends to zero, f_i decreases faster than f_r, and their ratio, which is proportional to R, tends also to zero as q and R tend to zero—the so-called "creeping motion" of flow. When f_i is negligible compared with f_r, then our force equilibrium becomes simply

$$\mathbf{f}_r = -\mathbf{f}_d \tag{60}$$

for every volume element within the flow system.

[6] "An Experimental Investigation of the Circumstances Which Determine Whether the Motion of Water Shall Be Direct or Sinuous and of the Law of Resistance in Parallel Channels," *Phil. Trans. Royal Soc. London*, Vol. 174 (1883), pp. 935–82; also *Papers on Mechanical and Physical Subjects* (Cambridge: University Press, 1901), Vol. II, pp. 51–105.

Now, if we agree to keep the flow velocities small enough so that the forces due to inertia are negligible and so that, therefore, equation (60) is valid, we can integrate f_r over all the microscopic volume elements of a macroscopic volume. The resultant will be $\mathbf{F}_r$, the resistive force acting upon the macroscopic element of volume. To perform this integration we observe that at every point f_r is directed parallel to the streamline, and, at least in the majority of cases, opposite to the direction of motion. Also, at each point f_r is proportional in magnitude to the velocity vector v.

Suppose the macroscopic flow is uniform and rectilinear. Then, if we choose an axis parallel to this macroscopic flow direction, we can resolve the microscopic velocity at every point into an axial component $v \cos \theta$ and a normal component $v \sin \theta$, where θ is the angle the streamline makes with this axis. Since the net fluid motion normal to the axis is zero, then there must be an equal number of normal components in all directions, so that they cancel one another completely.

The axial components, on the contrary, all have the same direction; and their average value, $\bar{v}$, over a macroscopic volume element is to be obtained by integration:

$$\bar{v} = \iiint\limits_{\epsilon \cdot \Delta V} \frac{v \cos \theta \cdot dV}{\epsilon \cdot \Delta V} = \frac{q}{\epsilon} . \tag{61}$$

The product $\epsilon \cdot \Delta V$ is the fraction of a macroscopic volume that is occupied by the fluid, and division by this quantity of the integral of equation (61) is necessary to obtain the average velocity. Otherwise we would have the *sum* of the axial components.

Now, bearing in mind that f_r is proportional to v, it, too, can be resolved into radial and axial components and integrated. For the radial components

$$\iiint\limits_{\epsilon \cdot \Delta V} f_r \sin \theta \cdot dV \propto \iiint\limits_{\epsilon \cdot \Delta V} v \sin \theta \cdot dV = 0 . \tag{62}$$

This leaves only the axial components whose algebraic sum or integral over the volume $\epsilon \cdot \Delta V$ is the macroscopic resistive force $\mathbf{F}_r$:

$$\iiint_{\epsilon \cdot \Delta V} f_r \cos \theta \cdot dV = \mathbf{F}_r \propto -\eta \cdot \frac{\mathbf{q}}{d^2} \cdot \epsilon \cdot \Delta V . \qquad (63)$$

Now, if we let $1/N$ be the constant of proportionality, we may write for the resistive force:

$$\mathbf{F}_r = -\frac{1}{N} \cdot \eta \cdot \frac{\mathbf{q}}{d^2} \cdot \epsilon \cdot \Delta V . \qquad (64)$$

If we divide this by $\rho\epsilon \cdot \Delta V$, we shall obtain

$$\frac{\mathbf{F}_r}{\rho\epsilon \cdot \Delta V} = -\frac{1}{N} \cdot \frac{\eta}{\rho} \cdot \frac{\mathbf{q}}{d^2} , \qquad (65)$$

which is the resistive force per unit of fluid mass.

On the macroscopic scale the two forces which act upon the fluid contained in a volume element ΔV are the driving force and the resistive force, and these must be equal in magnitude and opposite in direction, so that

$$\mathbf{F}_r = -\mathbf{F}_d . \qquad (66)$$

Also, the directions of both of these forces must be parallel to the macroscopic direction of flow, $\mathbf{F}_d$ in the direction of motion and $\mathbf{F}_r$ in the opposite direction.

We have already related $\mathbf{F}_r$ to the macroscopic motion of the fluid; let us now investigate the driving force, $\mathbf{F}_d$. Let us take a macroscopic element of volume consisting of a prism whose length is Δl and whose area of cross section is ΔA, oriented with its axis parallel to the direction of macroscopic flow. The driving force acting upon the fluid within this volume element will be the resultant of the surface forces upon its exterior and the body forces upon the mass within its interior. In this case the surface forces consist of normal stresses only, since the shear stresses are expended against the rigid framework of the medium throughout the interior of the body and are not transmitted to distances appreciably greater than the mean grain diameter d. Their effect is accordingly, on a macroscopic scale, similar to a body force and gives rise to the resistive force $\mathbf{F}_r$, which has already been evaluated. The normal stress acting upon the surface of the macroscopic volume element is the hydrostatic fluid pressure p. For the body force we have only the attraction due to gravity.

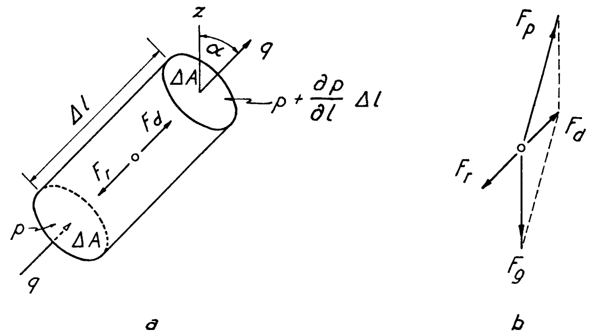

FIG. 8.—*a*, Forces acting upon macroscopic fluid volume element in direction parallel to flow; *b*, relation between forces parallel to flow and the total forces due to pressure gradient and gravity.

The total driving force must be the sum of the net force upon the volume element due to pressure and of the force due to gravitational attraction. Since the fluid is free to flow in any direction, this resultant must be in the direction of the fluid flow, and we need only consider the components of the other two forces in that direction.

For the component of the force due to pressure, let p be the pressure upon the upstream end of the prism, and $p + (\partial p/\partial l)\Delta l$ that upon the downstream end. We define the direction of the flow by the angle α which it makes from the upward directed vertical. Then in the direction of the flow α the component of the driving force due to pressure is

$$F_{p\alpha} = p\epsilon \cdot \Delta \mathrm{A} - \left(p + \frac{\partial p}{\partial l} \cdot \Delta l\right)\epsilon \cdot \Delta A = -\frac{\partial p}{\partial l} \cdot \epsilon \cdot \Delta V\ , \quad (67)$$

where $\epsilon \cdot \Delta A$ is the fraction of ΔA that is occupied by the fluid, and $\epsilon \cdot \Delta V$ the volume of fluid within the prism.

If we divide equation (67) by $\rho\epsilon \cdot \Delta V$, we obtain

$$\frac{F_{p\alpha}}{\rho \cdot \epsilon \cdot \Delta V} = -\frac{1}{\rho} \cdot \frac{\partial p}{\partial l}, \tag{68}$$

which is the component of the force per unit of mass of the fluid in the direction α produced by the change of macroscopic pressure in that direction. It is numerically equal to the rate of increase with distance of the pressure and oppositely directed. That is, the force due to pressure alone is always directed from regions where the pressure is higher toward those where it is lower.

The component in the direction α of the force exerted by gravity acting upon the fluid contained within the volume ΔV is

$$F_{g\alpha} = -g \cos \alpha \cdot \rho\epsilon \cdot \Delta V = -g \cdot \frac{\partial z}{\partial l} \cdot \rho\epsilon \cdot \Delta V \,. \tag{69}$$

The negative sign here is necessary to allow for the fact that for upward flow cos α is positive, while for downward flow it is negative. In the expression $\partial z/\partial l$, z is the vertical co-ordinate, or the elevation above the standard datum.

Now, adding equations (67) and (69) gives us the total driving force acting upon the fluid in the volume element ΔV,

$$\mathbf{F}_d = \left(-g \cdot \frac{\partial z}{\partial l} - \frac{1}{\rho} \cdot \frac{\partial p}{\partial l}\right) \rho\epsilon \cdot \Delta V \,; \tag{70}$$

and, if we divide this by $\rho\epsilon \cdot \Delta V$, we obtain

$$\frac{\mathbf{F}_d}{\rho\epsilon \cdot \Delta V} = -g \cdot \frac{\partial z}{\partial l} - \frac{1}{\rho} \cdot \frac{\partial p}{\partial l}, \tag{71}$$

which is the total driving force acting upon each unit of mass of the fluid.

If we turn now to equations (30) and (31) and differentiate them with respect to l, and then compare the results with equation (71), we shall find that

$$-g \cdot \frac{\partial z}{\partial l} - \frac{1}{\rho} \cdot \frac{\partial p}{\partial l} = -\frac{\partial \Phi}{\partial l}, \tag{72}$$

so that

$$\frac{\mathbf{F}_d}{\rho\epsilon \cdot \Delta V} = -\frac{\partial \Phi}{\partial l}\,. \tag{73}$$

Hence, the total driving force per unit of mass of the fluid is numerically equal to the rate of increase with distance of the fluid potential and is oppositely directed.

Since the sum of the macroscopic forces per unit of mass must be zero, then, by adding the expressions for these forces as given by equations (65) and (73), we obtain

$$-\frac{1}{N} \cdot \frac{\eta}{\rho} \cdot \frac{q}{d^2} - \frac{\partial \Phi}{\partial l} = 0\,,$$

which, when solved for q, gives

$$q = -Nd^2 \cdot \frac{\rho}{\eta} \cdot \frac{\partial \Phi}{\partial l} = -Nd^2 \cdot \frac{\rho}{\eta}\, g \cdot \frac{\partial h}{\partial l}\,. \tag{74}$$

When this is compared with Darcy's law as expressed by equation (34), it is clear that equations (34) and (74) are equivalent and that

$$K = Nd^2\rho \cdot \frac{1}{\eta} \cdot g\,, \tag{75}$$

so that the five factors to the right of equation (75) are evidently the quantities we originally set out to discover which were lumped together as the parameter K. Of these, all have been defined except the factor N, which we employed as a factor of proportionality when relating the resistive force to the fluid velocity. N is simply a dimensionless numerical coefficient whose value depends upon the geometrical shape of the internal structure of the medium through which the flow occurs. For two geometrically similar media the values of N would be the same; for dissimilar media, such as rounded versus angular grain shapes, the values of N would be different. The dimensions of N are therefore

$$[N] = [\text{angle}] = [LL^{-1}] = [L^0]\,. \tag{76}$$

Since its effect is determined solely by experiment, no more precise definition of N is required.

That the five quantities to the right in equation (75) are the correct ones, that they occur to their proper powers, and that no other essential quantities have been omitted may be demonstrated by a simple dimensional check. We rearrange equation (74) as follows:

$$\frac{q}{\frac{\partial h}{\partial l}} = -Nd^2\rho \cdot \frac{1}{\eta} \cdot g\,,$$

where the terms to be investigated are segregated to the right. If the equation is correct, then the dimensions of the right-hand term must be the same as those of the term to the left.

The dimensions of the separate factors are:

$$\left.\begin{aligned} [q] &= [LT^{-1}]\,, \\ \left[\frac{1}{\frac{\partial h}{\partial l}}\right] &= [L^0]\,, \\ [N] &= [L^0]\,, \\ [d^2] &= [L^2]\,, \\ [\rho] &= [ML^{-3}]\,, \\ \left[\frac{1}{\eta}\right] &= M^{-1}L^{+1}T^{+1}]\,, \\ [g] &= [LT^{-2}]\,. \end{aligned}\right\} \qquad (77)$$

Introducing these values into equation (74) gives the dimensional expression

$$\left[\frac{q}{\frac{\partial h}{\partial l}}\right] = \left[Nd^2\rho \cdot \frac{1}{\eta} \cdot g\right]\,, \qquad (78)$$

or

$$\left.\begin{aligned} [M^0L^{+1}T^{-1}] &= [L^0L^2M^{+1}L^{-3}M^{-1}L^{+1}T^{+1}L^{+1}T^{-2}] \\ &= [M^0L^{+1}T^{-1}]\,. \end{aligned}\right\} \qquad (79)$$

The fact that these dimensions balance is interpreted to mean that all the factors contained in K are given in the right-hand side of equation (78). Of these, only N might be further broken down into additional components, such as porosity and other geometrical parameters. In practice there is no need for such dissection, since N and d^2 are more conveniently lumped together as a single property of the medium. Consequently, we may regard the equations (74):

$$q = -Nd^2 \cdot \frac{\rho}{\eta} \cdot \frac{\partial \Phi}{\partial l} = -Nd^2 \cdot \frac{\rho}{\eta} \cdot g \cdot \frac{\partial h}{\partial l},$$

as equivalent and complete expressions of Darcy's law in the most general form we shall require. This tells us that, in addition to the factors investigated already, q varies directly as a factor N depending upon the internal shape of the medium, as the square of the grain size or other suitable length scale indicating coarseness, as the density of the fluid, and inversely as the viscosity of the fluid.

We come now to the problem of a more precise definition of *permeability* than the qualitative one given earlier. Of the five quantities N, d^2, ρ, $1/\eta$, and g of equation (74), the first two, N and d^2, are properties solely of the medium; the second two, ρ and η, are properties solely of the fluid; the last, g, is a property of the earth's gravitational field and may be taken, for present purposes, as constant. It would be possible to select from these five quantities any combination which includes the properties of the medium, N and d^2, and to make of this a single lumped parameter which could then be called the "coefficient of permeability." As will be discussed later, this is essentially what has been done already, so that numerous dimensionally unlike quantities, all called "permeability," are currently in use.

To avoid this sort of confusion, we ask what it is that we wish a coefficient of permeability to signify. We have already noted that the concept pertains to the facility with which a given rock or material transmits fluids. Persumably, then, permeability is to be taken as a property of the medium alone. If so, then a coefficient of permeability for a given medium should not change value when different fluids are employed. To satisfy this requirement we lump the factors Nd^2, depending upon the medium only, into the single factor k and then write Darcy's law as

$$q = -k \cdot \frac{\rho}{\eta} \cdot \frac{\partial \Phi}{\partial l}, \tag{80}$$

where k is the coefficient of permeability of the medium. For any given medium its value is obtained by an experiment analogous to that of Figure 1, where all quantities except k are measured by experiment, and then equation (80) is solved for k.

The dimensions of k are

$$[k] = [Nd^2] = [L^2]. \tag{81}$$

For purposes of mathematical analysis we are frequently concerned with the flow rate without regard to the parts played separately by the properties of the fluid and those of the medium. In this case it is more convenient to lump the factors k, ρ, and $1/\eta$ together into a single parameter σ, which then simplifies Darcy's law to

$$q = -\sigma \cdot \frac{\partial \Phi}{\partial l}, \tag{82}$$

a form which is physically, as well as mathematically, analogous to Ohm's law in electricity when applied to extended media. In the flow of electricity the factor analogous to σ is called the *specific electrical conductivity;* here we shall call σ the *specific fluid conductivity*. From its definition it is clear that the specific fluid conductivity is a property both of the medium and of the fluid. For a homogeneous liquid under isothermal conditions the density and viscosity are constant. In this case the specific conductivity will vary proportionately to the permeability of the medium.

RANGE OF VALIDITY OF DARCY'S LAW

We come now to the problem of the range of validity of Darcy's law, which states that the rate of flow increases linearly with the potential gradient. Over what values of q should this be true? This takes us back to our derivation of the value of the resistive force f_r

in terms of q in equation (58) and the equations preceding. It may be recalled that we were obliged to stipulate that the flow was to maintain kinematic similarity for different values of q and that we found we could satisfy this condition only by making the velocities so small that forces due to inertia became negligible. Under these conditions we deduced Darcy's law analytically.

What would happen now if we increased the velocity to where the forces of inertia are not negligible but are comparable in magnitude to the other forces? Since, in general, the inertial forces are not parallel to the streamlines but are randomly oriented, this can only produce distortion in the streamlines, causing crowding where the curvature is sharp. At still higher velocities eddies may also form in the wake of certain of the sand grains where the streamlines pull away from the grain surface. All this is still entirely within the range of strictly laminar flow and does not involve turbulence to any degree.

For uniform rectilinear macroscopic flow of a liquid the integral over a macroscopic volume of the forces due to inertia is zero, so that these can exercise no direct retarding effect upon the fluid motion. What they do, however, is produce distortion which increases the terms of the form $\partial^2 u/\partial y^2$, upon which the resistive forces depend. Since the forces due to inertia increase as the square of the velocity, this effect is nonlinear with respect to q; and for values of q for which the inertial forces are not negligible the resistance to the flow must increase at a progressively greater rate as q is increased. Hence, Darcy's law can only be valid for values of q small enough that the forces of inertia are negligible.

It is more convenient, because of the universality of the relationship, to refer this range of validity to the Reynolds' number R, defined by equation (59), since, in geometrically similar media, kinematically similar flow occurs at the same values of this quantity. In the present problem there are three values of Reynolds' number of particular significance: $R = 0$, corresponding to zero flow; $R = R^*$, the point at which inertial forces become effective; and $R = R_{\text{crit}}$, where the flow becomes turbulent. The values of R^* and R_{crit} for any particular geometrical framework can only be determined by experiment. Until now it has been almost the universal custom among the students of this sort of flow to assume without question that the departure from linearity of q with respect to $d\Phi/dl$, or its equivalent, was evidence of the beginning of turbulence. This error probably arises from the fact that most of the studies of the transition from linear to turbulent flow have been conducted upon flow through uniform straight tubes. In the flow of liquids through rectilinear tubes there exist no accelerations at all until the motion

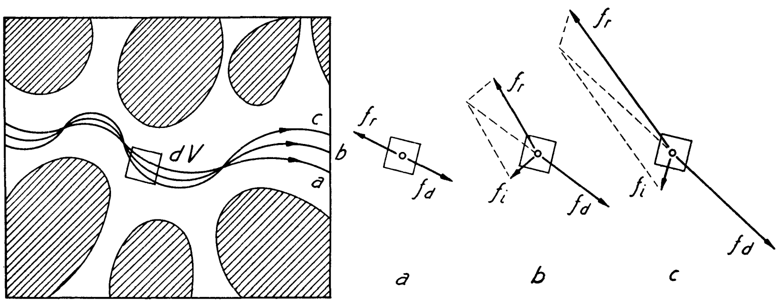

FIG. 9.—Schematic representation of the distortion of flowlines and force equilibria by increase of fluid velocity. *a*, *b*, and *c* represent flowlines and corresponding forces acting upon fixed volume element dV at successively higher velocities. Darcy's law is valid only for case *a*.

becomes unstable and turbulence sets in. Consequently, for this special case, R^* and R_{crit} coincide. In flow with curvilinear motion as well as acceleration parallel to the streamlines, the two values may be remote from one another.

For straight tubes the Reynolds' number is commonly defined in terms of the mean fluid velocity $\bar{v}$ and the radius r. Then the critical value is found experimentally to be

$$R_{\text{crit}} = \frac{\bar{v}r}{\frac{\eta}{\rho}} \cong 1{,}160\ .$$

For sediments we have defined the Reynolds' number to be

$$R = \frac{qd}{\frac{\eta}{\rho}}\,,$$

where q is the specific discharge and d the mean grain diameter. This obviously gives only an order of magnitude, because different sediments may depart widely from geometrical similarity. Lindquist,[7] employing a medium composed of uniform lead shot with a porosity of 38 per cent, found the value of R^* to be about 4. This was true for different experiments in which shot with diameters ranging from 1 to 5 mm. was employed. For values of R as high as 180 the flow was still laminar, so that for this system R_{crit} must be greater than 180.

For water at 20° C. the kinematic viscosity η/ρ is 0.01 c.g.s., so that through a sand composed of uniform spheres with 38 per cent pore space, the maximum specific discharge for which Darcy's law holds is obtained by solving the following equation for q:

$$\left.\begin{aligned} \frac{q_{\max} d}{\dfrac{\eta}{\rho}} &= R^* = 4 \,, \\ q_{\max} &= \frac{0.04}{d} \text{ cm/sec} \,. \end{aligned}\right\} \tag{83}$$

For a coarse sand with grain diameter of 0.1 cm. the maximum specific discharge for which Darcy's law would still be valid would be about 0.4 cm/sec. At this rate, from a sand 3 meters thick the flow of water into a well of 10 cm. radius would be at a rate of 7.5 liters (almost a gallon) per second, or about 4,000 barrels per day. For finer sands the limiting value of q would be correspondingly greater. Since, for such flow, the velocity diminishes rapidly with distance from the well, then it is clear that only exceptionally will rates of flow in ground water be encountered that are outside the upper range of validity of Darcy's law.

There is no apparent reason for supposing the existence of any lower limit to the range of validity of Darcy's law short of $R = 0$, corresponding to $q = 0$. Tests made by Fishel[8] of the flow of water

[7] Erik Lindquist, "On the Flow of Water through Porous Soil," *1er Congrès des grands barrages* (Stockholm, 1933), pp. 81–101.

[8] V. C. Fishel, "Further Tests of Permeability with Low Hydraulic Gradients," *Trans. Amer. Geophys. Union, 16th Ann. Meeting* (1935), pp. 499–503.

through sand established the validity of Darcy's law for values of q as small as about 4×10^{-8} cm/sec, corresponding to values of $\partial h/\partial l$ as low as 2 or 3 inch/mile, or about 6×10^{-5}.

DARCY'S LAW FOR GASES

While our investigation of the fluid potential was purposely made general enough to include both liquids and gases, our interest thus far has concerned itself principally with the interpretation of the experiment of Figure 1 in which the fluids employed have been liquids only. Where the permeability of different samples of materials is to be measured, there are many experimental advantages in the use of a gas, such as air, as the experimental fluid. It does not dissolve components of the specimen or react with it chemically. Before a gas can be used, however, we must extend the theory to include compressibility according to the gas laws.

Consider the steady flow of a gas through a tube of uniform section filled with sand or other uniformly permeable material under isothermal conditions. In this case, as the pressure along the tube changes, the specific volume of the gas will also change, so that the volume discharge q will no longer be the same over succeeding cross sections of the tube but will vary inversely as the pressure. Since there is no accumulation of gas in any part of the flow region, then the mass of gas flowing across each cross section must be constant. Then, for any given cross section, if q is the specific volume discharge for that section and ρ the density of the gas as it crosses the section, the mass of gas crossing unit area per unit of time must be

$$j = \rho q = \text{constant} , \tag{84}$$

where j is the *specific mass discharge*.

For any infinitesimal distance dl along the axis of the tube in the direction of the flow, the density ρ of the gas may be regarded as constant; and Darcy's law in its differential form,

$$q = -Nd^2 \cdot \frac{\rho}{\eta} \cdot \frac{\partial \Phi}{\partial l} ,$$

is just as valid for gases as for liquids. Then, if we multiply both sides by ρ, we obtain

$$\rho q = j = -Nd^2 \cdot \frac{\rho^2}{\eta} \cdot \frac{\partial \Phi}{\partial l}, \tag{85}$$

which is also valid for both gases and liquids.

For gases, however, ρ is no longer a constant but varies continuously with distance along the axis of the cylinder, the density and pressure being related by Boyle's law,

$$\frac{p}{\rho} = \frac{p_0}{\rho_0}, \qquad \text{or} \qquad \rho = p \cdot \frac{\rho_0}{p_0}, \tag{86}$$

where ρ_0 and p_0 are the density and pressure at the standard state of 1 atmosphere. Also, for this case we must employ the more general form of the fluid potential

$$\Phi = gz + \int_{p_0}^{p} \frac{dp}{\rho}$$

of equation (30), which, however, has the same derivative,

$$\frac{d\Phi}{dl} = g \cdot \frac{\partial z}{\partial l} + \frac{1}{\rho} \cdot \frac{\partial p}{\partial l}, \tag{87}$$

as the form valid for liquids only.

The viscosity of a gas, however, as predicted by the kinetic theory of gases and confirmed by experiment, is independent of pressure variations over a range of pressures comparable to that for which the gas laws are valid. Hence, η for a single gas at a constant temperature has the same value throughout.

Now, if we introduce equation (87) into (85), this gives us

$$j = -\frac{Nd^2}{\eta} \cdot \rho^2 \left(g \cdot \frac{\partial z}{\partial l} + \frac{1}{\rho} \cdot \frac{\partial p}{\partial l} \right). \tag{88}$$

The first term in the parentheses is the component of the force per unit of mass due to gravity; the second term is the corresponding force due to pressure. For gases under medium pressures ρ is small compared with unity, and the pressure gradient may be large, so that in all but exceptional cases the second term is so much larger than the first that no significant error will result if we neglect the term $g(\partial z/\partial l)$. With this simplification equation (88) reduces to

$$j = -\frac{Nd^2}{\eta} \cdot \rho \cdot \frac{\partial p}{\partial l}, \tag{89}$$

which then, by introducing the value for ρ given in equation (86), becomes

$$j = -\frac{Nd^2}{\eta} \cdot \frac{\rho_0}{p_0} \cdot p \cdot \frac{\partial p}{\partial l}. \tag{90}$$

Since, however,

$$p \cdot \frac{\partial p}{\partial l} = \frac{1}{2} \cdot \frac{\partial (p^2)}{\partial l}, \tag{91}$$

equation (90) can be written

$$j = -\frac{Nd^2}{\eta} \cdot \frac{\rho_0}{p_0} \cdot \frac{1}{2} \cdot \frac{\partial (p^2)}{\partial l}. \tag{92}$$

Then, since all other terms are constants already,

$$\frac{\partial (p^2)}{\partial l} = \text{constant}, \tag{93}$$

which tells us that the rate of decrease of the *square* of the pressure with distance in the direction of the flow is constant and that the specific mass discharge is proportional to this.

If we insert two pressure gauges into the system at a finite distance l apart (corresponding to the manometer tubes of Fig. 1), then

$$\frac{1}{2} \cdot \frac{\partial (p^2)}{\partial l} = \frac{p_2^2 - p_1^2}{2l} = \frac{p_2 + p_1}{2} \cdot \frac{p_2 - p_1}{l} = \bar{p} \cdot \frac{p_2 - p_1}{l}, \tag{94}$$

where p_1 and p_2 are the two pressures and $\bar{p}$ is their arithmetical average.

Introducing this into equation (92) reduces that equation to the form[9]

$$j = -\frac{Nd^2}{\eta} \cdot \frac{\rho_0}{p_0} \cdot \frac{\bar{p}(p_2 - p_1)}{l}, \tag{95}$$

for which every term is experimentally determinable except Nd^2. Then solving for Nd^2 gives the permeability k for the medium.

It need hardly be added that the value of the permeability of a given sand obtained in this manner should be in close agreement with the value obtained when a liquid is employed.

FLOW THROUGH EXTENSIVE MEDIA

So far we have considered only the case of uniform, rectilinear flow through a prismatic body of sand which is uniform and isotropic with respect to permeability. It remains now to determine whether the results obtained thus far are applicable to the general case of flow through extensive inhomogeneous media such as are encountered in field problems of ground-water flow. For this purpose we shall employ the macroscopic point of view exclusively. We shall also retain the assumption that the medium is isotropic with respect to permeability, although recognizing that in the case of some sedimentary rocks this assumption is not valid.

Our problem, therefore, is a *field* problem, and we need to introduce a few concepts which have become standard in dealing with such problems. When in any region of space there exists at each point a particular value of some given physical quantity—temperature, for example—that region is said to be a *field* in the quantity considered. If temperature were the quantity, the field would be a thermal field. If the quantity constituting the field is a *scalar*, that is, if it is definable by a single number, like temperature, then the field is a *scalar field*. If the quantity is a vector, like force or velocity, then the field is said to be a *vector field*. A still higher-order field exists for quantities, like stress, which in general are definable by nine independent quantities. This kind of field is a *tensor field*.

[9] This result has been derived earlier, using a different method from that given here, by Wyckoff, Botset, Muskat, and Reed ("Measurement of Permeability of Porous Media," *Bull. Amer. Assoc. Petrol. Geol.*, Vol. XVIII [1934], pp. 161–90).

The variation of most physical quantities in space is continuous with distance, so that the difference of the values of the same quantity at two near-by points decreases continuously as the distance between the points is made smaller and smaller, approaching zero as a limit. Regions in which the variation is continuous may, however, be bounded by surfaces of discontinuity.

In regions free from discontinuities, through each point in a scalar field a surface can be passed upon which all values of the scalar quantity are the same as that at the given point. This surface will then divide the region into two parts, in one of which all the values of the quantity considered will be higher, and in the other lower, than the values upon the surface itself. Such surfaces we may call *field surfaces*, and there will be one field surface passing through each point in the field. An isothermal surface is a field surface for a thermal field.

For vector fields in a region free from discontinuities we have *field lines*. A field line is the locus of a point which moves through a vector field in such a manner that at every point its direction of motion coincides with the direction of the vector at that point.

A *stationary field* is a field throughout which the values of the quantity of which the field consists remain invariant with time.

In a region free from discontinuities in the quantities considered, for every scalar field there exists an associated vector field, the relation between the two being obtained in the following manner:

Let Ω be any scalar quantity whose values are supposed known at every point in its field and whose field is stationary and continuous. At a point O in this field we erect x, y, and z axes. Through the origin let the field surface $\Omega = \Omega_0$ pass. To this we erect the surface normal n in the direction of increasing values of Ω. We let α, β, and γ be the angles that n makes with the positive direction of the x, y, and z axes, respectively. At a distance Δn from the origin the surface $\Omega_0 + \Delta\Omega$, sensibly parallel to the surface Ω_0, intersects the normal n. This same surface intersects the co-ordinate axes at distances Δx, Δy, and Δz from the origin.

Now what we are interested in finding is the rate of change with distance of the quantity Ω in the various directions in space away from the point O. In the direction n the average rate of change over the distance Δn is $\Delta\Omega/\Delta n$. The corresponding average rates along the three axes are $\Delta\Omega/\Delta x$, $\Delta\Omega/\Delta y$, and $\Delta\Omega/\Delta z$, respectively. But

$$\left.\begin{aligned} \Delta x &= \frac{\Delta n}{\cos \alpha}, \\ \Delta y &= \frac{\Delta n}{\cos \beta}, \\ \Delta z &= \frac{\Delta n}{\cos \gamma}, \end{aligned}\right\} \tag{96}$$

so that

$$\left.\begin{aligned} \frac{\Delta\Omega}{\Delta x} &= \frac{\Delta\Omega}{\Delta n} \cdot \cos \alpha , \\ \frac{\Delta\Omega}{\Delta y} &= \frac{\Delta\Omega}{\Delta n} \cdot \cos \beta , \\ \frac{\Delta\Omega}{\Delta z} &= \frac{\Delta\Omega}{\Delta n} \cdot \cos \gamma . \end{aligned}\right\} \tag{97}$$

Then, if we pass to the limits by letting Δn approach zero, equations (97) become

$$\left.\begin{aligned} \frac{\Delta\Omega}{\Delta x} \rightarrow \frac{\partial\Omega}{\partial x} &= \frac{\partial\Omega}{\partial n} \cdot \cos \alpha , \\ \frac{\Delta\Omega}{\Delta y} \rightarrow \frac{\partial\Omega}{\partial y} &= \frac{\partial\Omega}{\partial n} \cdot \cos \beta , \\ \frac{\Delta\Omega}{\Delta z} \rightarrow \frac{\Delta\Omega}{\partial z} &= \frac{\partial\Omega}{\partial n} \cdot \cos \gamma . \end{aligned}\right\} \tag{98}$$

The direction of the surface-normal n is determined by the field and, hence, fixed in space. The orientation of the x, y, and z axes, on the other hand, is arbitrary, so that, separately, the angles α, β, and γ may vary from 0° to 180°. Consequently, the ranges of the values of their cosines are

$$\left.\begin{aligned} -1 \gtreqless \cos \alpha \gtreqless 1 , \\ -1 \gtreqless \cos \beta \gtreqless 1 , \\ -1 \gtreqless \cos \gamma \gtreqless 1 . \end{aligned}\right\} \tag{99}$$

Now, if we take any line s making an angle θ with the direction of n, we are at liberty to orient the x axis along this line, making α and θ equal. Then

$$\frac{\partial \Omega}{\partial s} = \frac{\partial \Omega}{\partial x} = \frac{\partial \Omega}{\partial n} \cdot \cos \theta \equiv \text{grad } \Omega \cdot \cos \theta \,. \tag{100}$$

Here grad Ω (read "the gradient of Ω") is identical with $\partial\Omega/\partial n$. Since $\cos \theta$ is always equal to or less than unity, it follows that grad Ω, which is the rate of increase of Ω in the direction of the surface-

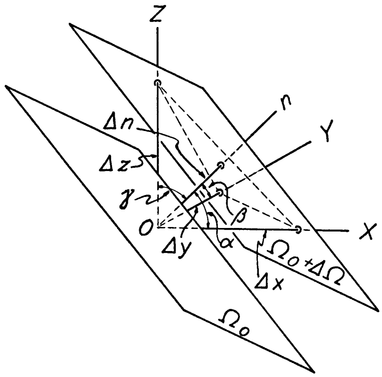

FIG. 10.—Space variation of continuous field of scalar quantity Ω.

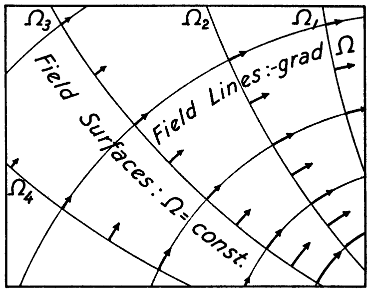

FIG. 11.—Relation between scalar field and derived vector field.

normal, is the maximum rate of increase in any direction. Also, since the rates of change in all other directions are related to grad Ω in the same manner as the components of a vector are related to the vector itself, then grad Ω may be taken to be a vector whose direction is parallel to n and whose magnitude is equal to the maximum rate of change of Ω with distance.

Thus, to every scalar field in any scalar quantity Ω there corresponds a vector field in the vector quantity grad Ω whose field lines are everywhere at right angles to the field surfaces Ω = a constant, and directed from regions where Ω is smaller toward those where it is larger.

The physical dimensions of the vector quantity grad Ω are

$$[\text{grad } \Omega] = \left[\frac{\Omega}{\text{length}}\right] , \tag{101}$$

which may represent a force, a velocity, or other vector quantity, depending upon the nature of the scalar Ω. Also, as many different kinds of scalar and vector fields may coexist in the same region of space as there are scalar and vector quantities at each point in that space.

APPLICATION TO FLUID FLOW

With this background, let us now return to consider a field in which ground water or other fluid is our particular concern. We shall suppose that all fields are stationary in order not to be obliged to consider changes of the field with time. Even when such changes are imposed, we shall consider only the steady states that are established before and after the change, and shall avoid the *transient* stage during which the changes take place. In many cases where there are more or less continuous fluctuations governed by irregularities of local precipitation and other climatic factors, we shall be content to consider the average state of the system over a time period of a year or more.

Among the scalar quantities whose fields are of interest in the study of fluid motion are those of the hydrostatic pressure p, the gravity potential U, and the fluid potential Φ. Each of these quantities has its separate family of field surfaces, normal to which is the respective family of field lines corresponding to the vector fields of the quantities grad p, grad U, and grad Φ. In addition to these three derived vector fields we have that of the flow of the fluid itself.

Our problem now is to investigate these several kinds of fields in order to learn the manner in which they influence the flow of a fluid through a permeable medium in an extensive region of space. Taking the most obvious vector field first, let us consider that of the flow itself. By injecting filaments of dye or other visible markers at specific points, we find that the fluid passing through each point follows a definite path, which we shall speak of as a *flowline* or *streamline*. The family of all flowlines is evidently the system of field lines of a vector field. The vector itself is still to be determined.

The flow vector can be determined if we delineate a small plane surface element of area ΔS whose normal makes the angle θ with the lines of flow. Then the projection ΔS_n of this surface element upon a plane perpendicular to the flowlines is given by

$$\Delta S_n = \Delta S \cdot \cos \theta \, . \tag{102}$$

Across ΔS_n the volume of fluid discharging per second is

$$q \cdot \Delta S_n = q \cdot \Delta S \cdot \cos \theta = q_n \cdot \Delta S \, , \tag{103}$$

where

$$q_n = q \cos \theta \, . \tag{104}$$

Thus the total discharge across a surface ΔS due to the specific discharge q making an angle θ with the surface normal is exactly the same as the discharge across the same surface due to a specific dis-

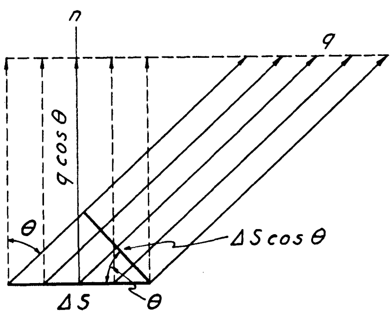

FIG. 12.—Vector properties of the specific volume discharge q

charge q_n parallel to the surface normal, where q_n bears the relation to q of a vector component to the vector itself. Consequently, the specific discharge q, which has a definite magnitude and direction at every point, is a vector, and its components in all other directions are obtainable by the usual methods of vector resolution. The vector $\mathbf{q}$ is everywhere tangential to the flowlines and so is a fundamental vector of the field of flow.

Another equally important flow vector, however, is the specific mass discharge $\mathbf{j}$. In direction this coincides at every point with $\mathbf{q}$, but its magnitude is $\rho\mathbf{q}$ where ρ is constant for liquids but varies along the flowlines for a gas. Thus the vector fields for the quantities $\mathbf{q}$ and $\mathbf{j}$ have identical flowlines but different magnitudes at corresponding points.

A fundamental analytical property of the vector **j** is discovered if we erect co-ordinate axes in the field of flow and choose a small macroscopic volume element whose lengths of sides are Δx, Δy, and Δz. We resolve the vector **j** into components j_x, j_y, and j_z parallel to the axes. We then determine the net mass of fluid flowing *out of*

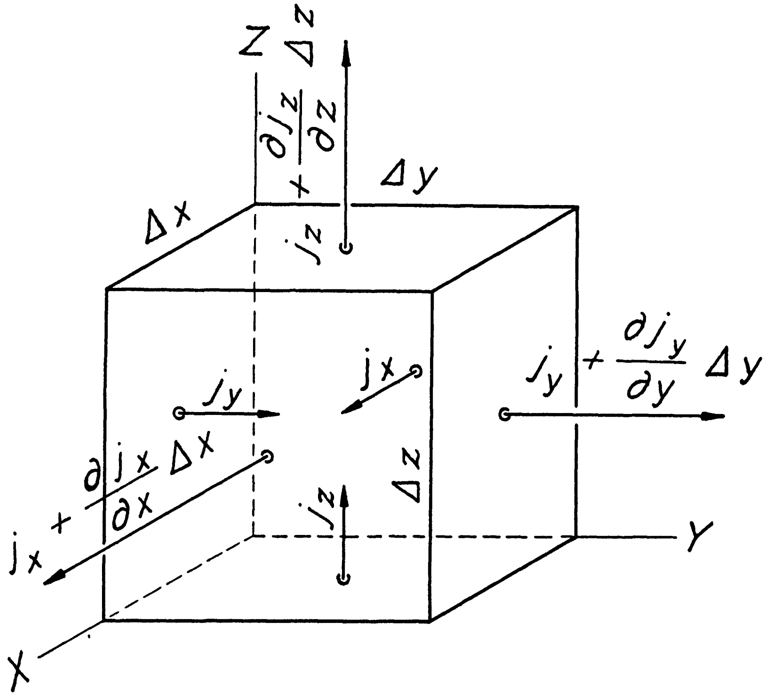

FIG. 13.—Mass discharge across the faces of macroscopic volume element

the volume element ΔV per unit of time. To do this we investigate the outward discharge across each of the six faces of the volume element. If we let J_x, J_y, and J_z be the total outward mass discharge across the x, the y, and the z faces, respectively, we shall have

$$J_x = -j_x \Delta y \cdot \Delta z + \left(j_x + \frac{\partial j_x}{\partial x} \cdot \Delta x\right) \Delta y \cdot \Delta z = \frac{\partial j_x}{\partial x} \cdot \Delta V \,, \quad (105)$$

$$J_y = -j_y \Delta z \cdot \Delta x + \left(j_y + \frac{\partial j_y}{\partial y} \cdot \Delta y\right) \Delta z \cdot \Delta x = \frac{\partial j_y}{\partial y} \cdot \Delta V \,, \quad (106)$$

$$J_z = -j_z \Delta x \cdot \Delta y + \left(j_z + \frac{\partial j_z}{\partial z} \cdot \Delta z\right) \Delta x \cdot \Delta y = \frac{\partial j_z}{\partial z} \cdot \Delta V \,. \quad (107)$$

Then the net outward mass discharge per unit of volume will be

$$\frac{J_x + J_y + J_z}{\Delta V} = \frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} + \frac{\partial j_z}{\partial z} . \tag{108}$$

Since mass is a conservative property of matter, the net mass outflow per unit of time per unit of volume must equal the loss of mass in that time by the volume. The mass content of unit volume is

$$m = \rho\epsilon , \tag{109}$$

so that the mass lost per unit volume in unit time must be

$$-\frac{dm}{dt} = -\epsilon \cdot \frac{\partial \rho}{\partial t} . \tag{110}$$

Equating (110) with (108) gives

$$\frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} + \frac{\partial j_z}{\partial z} = -\epsilon \cdot \frac{\partial \rho}{\partial t} , \tag{111}$$

which is known as the *equation of continuity* for fluid motion. It states the conditions that the flow vectors must satisfy at all points if the principle of the conservation of matter is not to be violated.

But for stationary flow $\partial\rho/\partial t = 0$. Then for this condition we have for both liquids and gases

$$\frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} + \frac{\partial j_z}{\partial z} = \frac{\partial(\rho q_x)}{\partial x} + \frac{\partial(\rho q_y)}{\partial y} + \frac{\partial(\rho q_z)}{\partial z} = 0 . \tag{112}$$

For an incompressible fluid the density is constant and so may be factored out of equation (112), leaving

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} = 0 \tag{113}$$

as a special form of the equation of continuity valid for liquids only.

It is customary, when dealing with liquids, to measure the volume rather than the mass; and, since in what follows our concern will be almost exclusively with the flow of liquids, we shall employ the form given in equation (113) using **q** as our flow vector, unless cases arise which require the more general form of equation (112). Equation (113) may be thought of as expressing the constancy of the fluid volume, or better, of defining the properties of the flow if the condition of the constancy of volume is not to be violated. This condition makes it impossible for the flow vector to change capriciously or discontinuously with distance, and so justifies our earlier stipulation of continuity in our vector and scalar fields except across bounding surfaces.

Another consequence of the condition of incompressibility of fundamental importance with regard to the motion of ground water is this: If we take a closed surface S enclosing a finite volume V within the field of flow, then across an element ΔS of the surface the outward directed flow per unit of time will be

$$\Delta Q = q \cos \theta \cdot \Delta S = q_n \cdot \Delta S \,, \tag{114}$$

where θ is the angle between the flow vector $\mathbf{q}$ and n, the outward directed surface normal. Then the net outward discharge over the whole surface will be

$$Q = \int\!\!\int_S q_n \cdot dS = 0 \,. \tag{115}$$

This becomes especially useful when we form the surface S in the following manner: Upon a surface which is normal to the flowlines we draw a closed curve enclosing an area A_1. Through each point of this curve a particular flowline passes, and the *ensemble* of such contiguous flowlines forms a sort of sleeve or *stream tube.* Some distance downstream from the surface A_1, we take another surface orthogonal to the flowlines and upon this the streamlines forming the walls of the stream tube circumscribe an area A_2. These two ends, A_1 and A_2, and the side walls of the tube in the intervening section then, together, form a closed surface, S. Over the area A_1 the angle θ is $180°$, and its cosine -1; over A_2, θ is 0 and its cosine $+1$; over the side walls θ is $90°$ and its cosine 0.

Then for this case the integral (115) over the surface S reduces to

$$Q = \int\!\!\int_{A_1} -q \cdot dS + \int\!\!\int_{A_2} q \cdot dS + 0 = 0 \,. \tag{116}$$

Furthermore, if the tube is made narrow enough, the magnitude of **q** over the cross sections A may be considered constant, in which case equation (116) simplifies to

$$-q_1A_1 = +q_2A_2 ,$$

or, in magnitude,

$$q_1A_1 = q_2A_2 = qA = \text{constant} , \qquad (117)$$

which tells us that for an incompressible fluid in steady flow the discharge across each cross section of any given stream tube is a constant. Also, there is a reciprocal relationship between specific discharge and area of cross section, the former tending to infinity as the latter tends to zero, and vice versa.

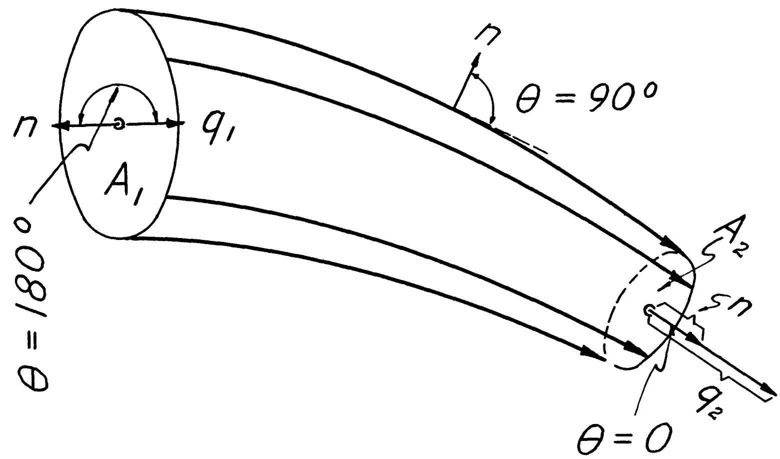

FIG. 14.—Net discharge over the surfaces of a stream tube

Since infinite specific discharge is precluded by the increase of resistance with velocity of flow, then a convergence of a stream tube to zero without violating the condition of incompressibility or the principle of the conservation of matter is impossible. Consequently, except at certain "singular points" characterized by a velocity of zero, no two noncontiguous streamlines can ever intersect or come together, since we can always choose a stream tube in such a manner that these two lines occur diametrically opposite each other, so that in order for them to come together the tube would have to converge to zero area of cross section.

Since the discharge along a stream tube is unidirectional and constant in amount, it follows that no stream tube of finite cross section can terminate except upon boundaries of the flow region, one end of each tube acting as an inlet and the other as an outlet. If the flowlines are radial with respect to an inlet or outlet region, then away from this region the cross section of each tube increases without limit as the distance increases, and the velocity of flow proportionately diminishes, tending to the limit zero. While this limit may not be reached, the motion becomes imperceptibly small, and we may think of the stream tube as having terminated by dispersion.

The intake end of a stream tube we may speak of as its *source*, and the outlet end as its *sink*. This usage is slightly different from that of standard potential theory in which a source is a region where the field is, so to speak, created, and a sink a region where it is annihilated. In that sense, if wheat in its granular form were the flow medium, then a wheat field would constitute an absolute source of this medium, and a flour mill an absolute sink. At those points the amounts entering and leaving would not be the same, whereas between those points the total quantity would be conserved, and the flow said to be *solenoidal*.

By our usage, an example of a sink in ground-water flow would be a well toward which the flowlines converge; an example of a source would be a lake from which water flows into the ground. Between source and sink the quantity of fluid is conserved, and the character of the flow is solenoidal.

Let us now consider the fields of the aforementioned scalar quantities: the pressure p, the gravity potential U, and the fluid potential Φ.

Take the pressure first. To every point within the fluid there will correspond a particular value of the hydrostatic pressure p. Through each point there passes one of a family of field surfaces, p = constant, to be known as "surfaces of equal pressure," or "*isobaric* surfaces." The corresponding vector field is made up of the vectors grad p which generate a family of field lines everywhere normal to the isobaric surfaces.

By reasoning analogous to that which led to equations (70) and (71), if we take a small fluid prism with its axis parallel to the normal n of the isobaric surface erected in the direction of increasing pressure, then the net force acting upon it in the direction n is

$$\mathbf{f}_p = -\frac{\partial p}{\partial n} \cdot \Delta V = -\text{grad } p \cdot \Delta V . \tag{118}$$

The force per unit of volume of the fluid is

$$\frac{\mathbf{f}_p}{\Delta V} = -\text{grad } p .$$

This is the total pressure force, of which that in any other direction is only a component. Consequently, at every point within the fluid there acts, owing to the pressure gradient, a force per unit of fluid volume which is numerically equal to grad p and oppositely directed. Hence, associated with the scalar field in p there exists an orthogonal family of lines of force per unit of volume, $-$grad p.

The force due to pressure per unit of mass is

$$\frac{\mathbf{f}_p}{\rho \cdot \Delta V} = -\frac{1}{\rho} \cdot \text{grad } p . \tag{119}$$

Now let us consider the gravity potential U. This quantity we shall define as equal to the gravity potential energy of unit mass with respect to the standard datum. Then, regarding g as constant,

$$U = gz , \tag{120}$$

where g is merely the scalar magnitude of gravity acceleration and z, as previously, is the vertical elevation.

The field surfaces, U = constant, known as *gravity equipotential surfaces*, obviously correspond to z = constant and are level surfaces. The field lines normal to these are parallel to the direction of gravity and are given by

$$\mathbf{g} = -\text{grad } U , \tag{121}$$

where in this case the vector $\mathbf{g}$ signifies the force per unit of mass, or acceleration due to gravity, both in magnitude and direction.

Finally, for the fluid potential Φ we have the family of *fluid equipotential* surfaces and its associated field lines,

$$\frac{\mathbf{f}_d}{m} = -\text{grad}\ \Phi\ , \tag{122}$$

where $-$grad Φ represents the force per unit of mass acting upon the fluid due to the gradient of the fluid potential Φ. This family of lines of force is normal to the fluid equipotential surfaces and oppositely directed from grad Φ.

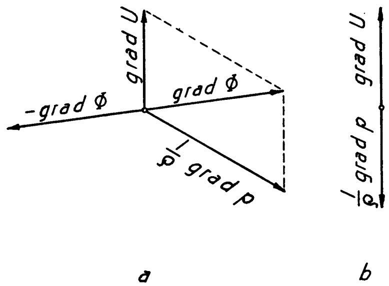

FIG. 15.—*a*, Vector relations between the gradients of the scalar fields of gravity potential, pressure, and fluid potential; *b*, condition for static equilibrium.

As we have seen already, the fluid potential is the sum of the work done against both gravity and pressure and is, in fact, given by

$$\Phi = U + \int_{p_0}^{p} \frac{dp}{\rho}\ , \tag{123}$$

where the gravity potential U is employed in place of gz. Then, since $-$grad Φ is the total force acting per unit of mass, it must be equal to the vector sum of the other two component forces:

$$-\text{grad}\ \Phi = -\text{grad}\ U - \frac{1}{\rho} \cdot \text{grad}\ p\ . \tag{124}$$

The other two fields, being thus combined into the single field $-$grad Φ, may therefore be dropped from further consideration except for special purposes which may arise. The surfaces $U =$ constant are horizontal and the direction $-$grad U is always downward; the vectors $-\text{grad } p$ and $-\text{grad } \Phi$, however, may have any direction whatever. Only in the special case that

$$-\text{grad } \Phi = 0$$

do we have

$$-\frac{1}{\rho} \cdot \text{grad } p = -(-\text{grad } U) , \tag{125}$$

for which hydrostatic equilibrium prevails.

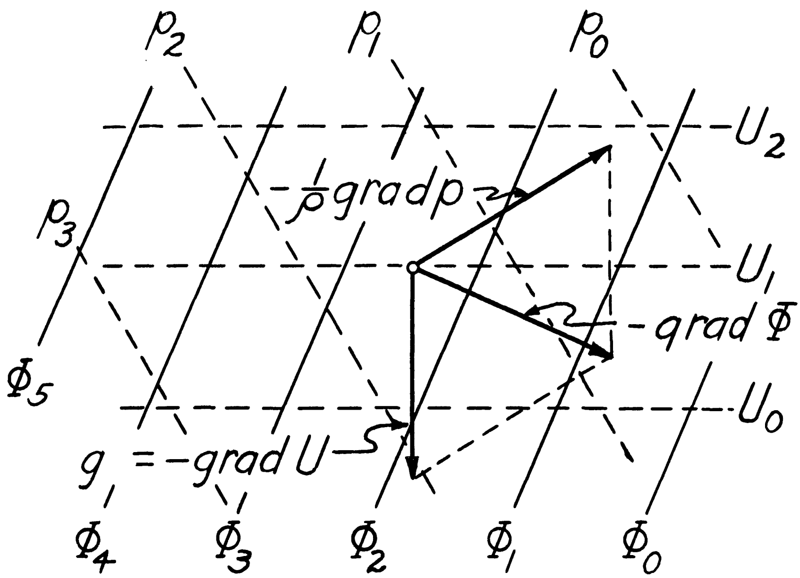

FIG. 16.—Relation between superimposed scalar fields of p, U, and Φ, and their corresponding negative gradients.

We are now in a position to elucidate the properties of the potential Φ in yet another manner. Within a flow field of a general kind, starting from an initial point P_0, let us carry, by a frictionless process (infinitely slow motion), an element of the fluid of unit mass along any arbitrary path A to a final position P. Under these conditions, owing to the field there will act upon the mass at each point a force $\mathbf{f}_d$, and to move the fluid we shall have to exert upon it an equal and opposite force $-\mathbf{f}_d$. The work required to traverse the distance ds of the path will be

$$dw = f_d \cdot ds \cdot \cos\theta , \tag{126}$$

where the quantity to the right is the product of the displacement by the component of force in that direction, and θ is the angle between the direction of the displacement and of the force $-\mathbf{f}_d$. But

$$f_d \cdot ds \cdot \cos\theta = \left(\frac{\partial U}{\partial s} + \frac{1}{\rho} \cdot \frac{\partial p}{\partial s}\right) ds = dw . \tag{127}$$

Then the total work required to carry the mass along the path A from P_0 to P is

$$w_A = \int dw = \int_A \left(\frac{\partial U}{\partial s} + \frac{1}{\rho} \cdot \frac{\partial p}{\partial s}\right) ds , \tag{128}$$

where $\int_A$ signifies the line-integral along the path A.

Next, suppose we carry a unit mass from P_0 to P along a different path B. The work required to do this will be

$$w_B = \int_B \left(\frac{\partial U}{\partial s} + \frac{1}{\rho} \cdot \frac{\partial p}{\partial s}\right) ds . \tag{129}$$

Then, if w_A is equal to w_B for all possible paths A and B, we may characterize the fluid at the point P by its possession of a uniquely determinable amount of mechanical energy Φ per unit of mass defined by

$$\Phi = \Phi_0 + w = \Phi_0 \int \left(\frac{\partial U}{\partial s} + \frac{1}{\rho} \frac{\partial p}{\partial s}\right) ds , \tag{130}$$

where Φ_0 is the corresponding quantity at the point P_0. If, on the contrary, the value of the line integral from P_0 to P is different for different paths, then $w_A \neq w_B$, and the quantity Φ defined by equation (130) is not unique. In the first case, where the line integral is unique, we say the field has a potential; in the second case, no potential is possible.

The most rigid criterion for the existence of a potential is obtained if we traverse a closed path, proceeding from P_0 to P by the path A and returning by B. Then the total work along the closed path is

$$w = w_A + w_B = \oint \left(\frac{\partial U}{\partial s} + \frac{1}{\rho} \cdot \frac{\partial p}{\partial s} \right) ds \, . \tag{131}$$

If this line-integral around all possible closed paths is equal to zero, a potential exists, and to each point in the field there corresponds a uniquely determinable value of the quantity Φ. If this integral around a closed path is not zero, no potential is possible, and there will be a circulatory component of the fluid flow around this path. This does not imply perpetual motion but indicates the presence of a heat engine, where heat is being added at one temperature in one part of the circuit and withdrawn at a lower temperature at another.

Interpreting equation (131) mechanically, we know already that work against gravity is independent of the path, so that

$$\oint \frac{\partial U}{\partial s} \cdot ds = 0 \, . \tag{132}$$

Then, if we are to have a fluid potential, it is also necessary that

$$\oint \frac{1}{\rho} \cdot \frac{\partial p}{\partial s} \cdot ds = 0 \, , \tag{133}$$

which is satisfied completely when the density ρ is uniquely determined for each value of the pressure. This condition is satisfied for incompressible fluids for which the density is constant and for isothermal or adiabatic transformations in gases. For liquids under a temperature gradient there actually may be a slight rotational component due to changes of density with temperature, but in ground water this is ordinarily of a negligible magnitude compared with the potential forces acting.

Equation (130) transforms to our earlier definition of fluid potential derived from thermodynamic reasoning when P_0 is situated at a point $z = 0$ with a pressure of p_0, and $\Phi_0 = 0$. Then at point P

$$\Phi = \int_{U_0}^{U} dU + \int_{p_0}^{p} \frac{dp}{\rho} = gz + \int_{p_0}^{p} \frac{dp}{\rho} \, , \tag{134}$$

where the right-hand quantity is our original definition of the fluid potential.

Potentials are classified according to the vector quantity to which they are related. Thus, a scalar quantity, the space derivative of whose field is a force, is said to be a *force potential;* if the space derivative is a velocity, the quantity is a *velocity potential.* Since the space derivative of the quantity Φ is a force, Φ must be a force potential.

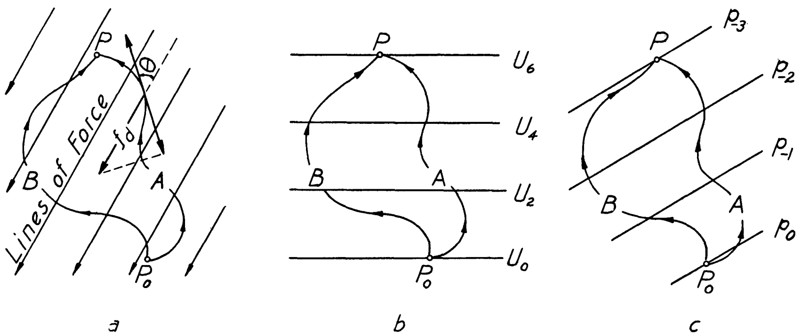

FIG. 17.—*a*, Fluid potential as a line integral in a field of force; *b* and *c*, the corresponding line integrals in the scalar fields of p and U.

RELATION BETWEEN THE POTENTIAL FIELD AND THE FLOW FIELD

So far we have investigated separately the properties of the fields of flow whose vectors are $\mathbf{q}$ and $\mathbf{j}$, respectively, and of the scalar field of the fluid potential and its associated force field $-\text{grad}\ \Phi$. These fields coexist in the same region of space, and our problem now is to relate them one to the other. This is accomplished directly by means of Darcy's law, of which we may employ either of the equivalent forms

$$\left.\begin{aligned} q_s &= -\sigma \cdot \frac{\partial \Phi}{\partial s}\,, \\ \dot{j}_s &= -\rho\sigma \cdot \frac{\partial \Phi}{\partial s}\,, \end{aligned}\right\} \qquad (135)$$

where q_s and j_s are the components in the direction s of the specific volume discharge and mass discharge, respectively. The total flow vectors, of which these are the components, are likewise given by

$$\left.\begin{aligned} \mathbf{q} &= -\sigma\ \text{grad}\ \Phi\,, \\ \mathbf{j} &= -\rho\sigma\ \text{grad}\ \Phi\,, \end{aligned}\right\} \qquad (135a)$$

which may be taken as equivalent general statements of Darcy's law. In these the vectors **q** and **j** are everywhere parallel to each other and to the streamlines, and the magnitude of **j** is equal to that of **q** multiplied by the factor ρ.

The relation of the vectors **q** and **j** to that of $-\text{grad}\ \Phi$ can best be seen by recalling that $\sigma = k\rho/\eta$ and that each of these quantities is a possible variable in the field of flow. If the value of k changes from point to point, the medium is inhomogeneous; and if it varies at the same point for flow in different directions, it is anisotropic.

For isotropic media the directions of both **q** and **j** will be the same as that of $-\text{grad}\ \Phi$; and their magnitudes will be σ and $\rho\sigma$, respectively, times that of $-\text{grad}\ \Phi$, and the flowlines will be normal to the equipotential surfaces. For anisotropic conditions the flowlines will, in general, be somewhat oblique to the direction of $-\text{grad}\ \Phi$.

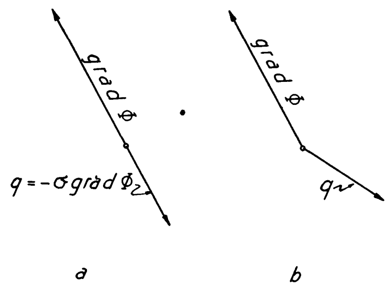

FIG. 18.—Vector representation of Darcy's law: *a*, for isotropic; *b*, for anisotropic media.

For the general fluid, ρ will vary as a function of the pressure, and η as a function of the temperature, and both as a function of distance in the field of flow, though for liquids at constant temperature this variation becomes negligibly small. Since we have agreed to consider only isotropic media, the flowlines with which we deal will be everywhere parallel to $-\text{grad}\ \Phi$ and will form an orthogonal system with the family of equipotential surfaces $\Phi = \text{constant}$. Homogeneity, however, cannot be assumed, for in rocks the value of k varies from zero for impermeable materials to infinity for large open spaces. For a fluid of constant density and viscosity flowing through rocks with similar internal structures, σ varies as k, or as the square of the grain diameter or other characteristic length. Thus, for a range of grain diameters from 10^{-3} to 1 cm., corresponding to the range of sediments from silt to coarse gravel, the increase of permeability and conductivity is of the order of a million-fold. It is highly important, therefore, that we develop our theory so as to take large differences of permeability into account.

Attention may here be directed to an important consequence of the wide variation of σ, which will be discussed in more detail later. There is an extensive employment among authors on this subject of an expression of the form

$$\mathbf{q} = -\text{grad}\,(\sigma\Phi)\,, \tag{136}$$

which is stated to be Darcy's law. Comparison of this with equation (135*a*) shows that, in order for this to be true, grad $(\sigma\Phi)$ must be equal to σ grad Φ. Actually, however, if we expand grad $(\sigma\Phi)$, we obtain

$$\text{grad}\,(\sigma\Phi) = \sigma\,\text{grad}\,\Phi + \Phi\,\text{grad}\,\sigma\,,$$

from which it is clear that equations (135*a*) and (136) can only be equivalent provided Φ grad $\sigma = 0$. This last condition is satisfied only in case $\sigma = k\rho/\eta =$ constant. Hence, for inhomogeneous media or for fluids of variable density or viscosity—that is to say, for general conditions—equation (136) must be ruled out as an expression of Darcy's law, on the grounds of being physically erroneous.

REFRACTION OF FLOWLINES ACROSS BOUNDARIES BETWEEN DIFFERENT MEDIA

With this point clear let us now proceed to investigate the flow of a liquid through inhomogeneous media. Consider first the case of flow across the boundary between regions of markedly different conductivities. Let σ_1 be the conductivity of the first region, and σ_2 that of the second. Let the interface be a plane surface whose normals into the two regions are n_1 and n_2. Let θ_1 and θ_2 be the angles the flowlines make with the normals in the first and second regions, respectively; and $\mathbf{q}_1$ and $\mathbf{q}_2$, the corresponding flow vectors.

At the boundary between these two regions two independent physical requirements must be satisfied simultaneously. The principle of the conservation of matter requires that the fluid leaving one region per unit of time must equal that entering the other. Since this involves only normal components of the flow

$$-(q_n)_1 = +(q_n)_2\,, \tag{137}$$

where $(q_n)_1$ and $(q_n)_2$ signify the normal components of flow *away from* the boundary in the two regions, respectively.

The principle of the conservation of energy requires that the line integral of the potential around a closed path, starting at a point on the boundary, proceeding a distance Δs along the boundary inside

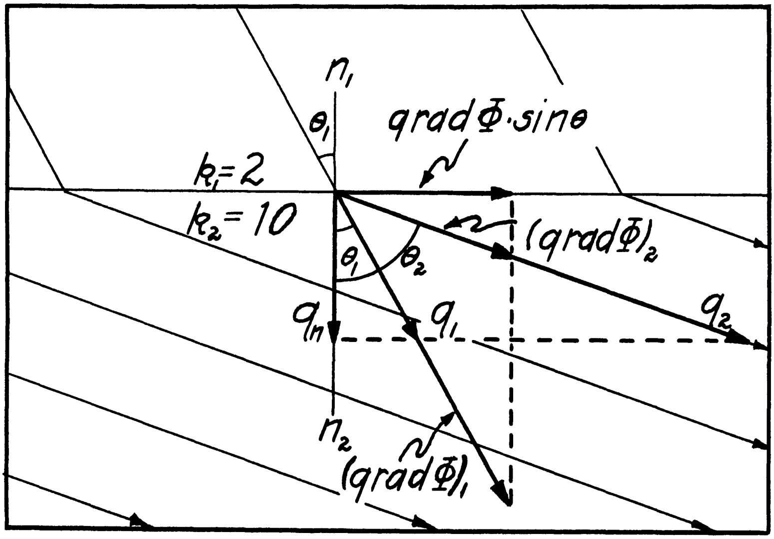

FIG. 19.—The refraction of flowlines at the interface between media of different permeabilities. The value of k_2/k_1 is here taken to be 5. The refraction obeys a tangent law.

region 2 and returning along a parallel path just inside region 1, must be zero. Hence,

$$\left(\frac{\partial \Phi}{\partial s}\right)_1 = \left(\frac{\partial \Phi}{\partial s}\right)_2 . \qquad (138)$$

If we carry out these operations in the plane containing the two surface normals n_1 and n_2 and the two flow vectors $\mathbf{q}_1$ and $\mathbf{q}_2$, and express the rates of flow in terms of the potential gradients and the conductivities, we shall have for the normal components of flow

$$\sigma_1(\text{grad } \Phi)_1 \cos \theta_1 = \sigma_2(\text{grad } \Phi)_2 \cos \theta_2 ; \qquad (139)$$

and for the tangential components of the potential gradient,

$$(\text{grad } \Phi)_1 \sin \theta_1 = (\text{grad } \Phi)_2 \sin \theta_2 . \qquad (140)$$

Dividing equation (140) by (139) then gives us

$$\frac{\sin \theta_1}{\sigma_1 \cos \theta_1} = \frac{\sin \theta_2}{\sigma_2 \cos \theta_2},$$

or

$$\frac{\tan \theta_1}{\tan \theta_2} = \frac{\sigma_1}{\sigma_2} = \frac{\dfrac{k_1 \rho_1}{\eta_1}}{\dfrac{k_2 \rho_2}{\eta_2}} = \frac{k_1}{k_2}, \tag{141}$$

which may be thought of as the law of refraction of the flowlines in fluid flow. At the boundary between media of different permeabilities both the flowlines and the equipotential surfaces change direc-

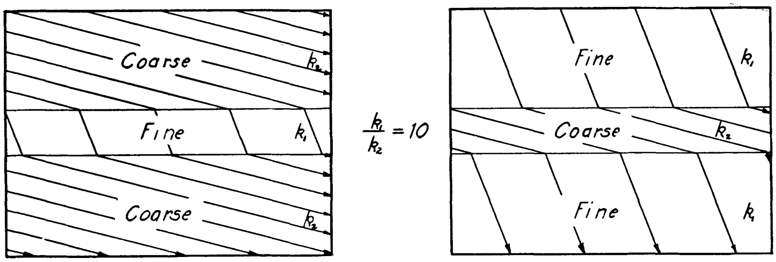

FIG. 20.—Refraction across layers of coarse and fine sand with a permeability ratio of 10.

tion abruptly. This resembles the refraction of light except that the latter obeys a sine law instead of a tangent law of refraction. It should be remarked, however, that this law of refraction is only true provided both media are completely filled with the same fluid.

The two cases of greatest importance are those for which, while one of the conductivities, say σ_1, remains finite, the other approaches either zero or infinity. The consequences of this are best brought out by equations (139) and (140). When $\sigma_2 \rightarrow 0$, equation (139), becomes

$$\sigma_1 (\text{grad } \Phi)_1 \cos \theta_1 = 0 . \tag{142}$$

When $\sigma_2 \rightarrow \infty$, $(\text{grad } \Phi)_2 \rightarrow 0$ and equation (140) becomes

$$(\text{grad } \Phi)_1 \sin \theta_1 = 0 . \tag{143}$$

Since both σ_1 and $(\text{grad } \Phi)_1$ are finite, then in the first case $\cos \theta_1 = 0$, and in the second, $\sin \theta_1 = 0$, corresponding to $\theta_1 = 90°$ and $\theta_1 = 0$ in the first and second cases, respectively.

Hence, in a region of finite permeability, equipotential surfaces terminate perpendicularly upon all impermeable boundaries; whereas upon permeable boundaries to regions of infinite conductivities (large voids) filled with the same fluid, all flowlines terminate perpendicularly.

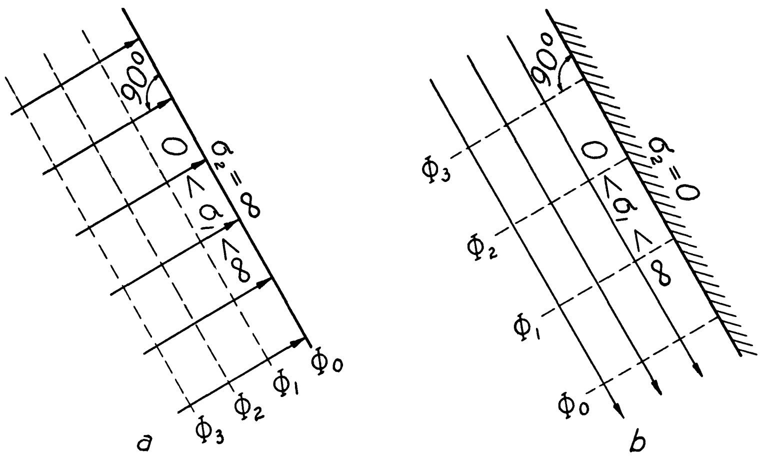

FIG. 21.—*a*, Flow from region of finite into one of infinite permeability; *b*, flowlines and equipotential surfaces along impermeable boundaries.

That this should be so becomes more obvious when we consider that upon impermeable boundaries the normal components of the flow are zero, so that the total flow to which the equipotential surfaces are perpendicular is the tangential component. For the case of flow into a fluid filled void or basin, the rate of flow is finite, but the conductivity is essentially infinite. Then, since $\mathbf{q} = -\sigma \text{ grad } \Phi$, it follows that grad Φ must tend to zero as σ increases without limit. Such a region approaches a condition Φ = constant, or becomes essentially an equipotential region. Since the tangential components of the potential gradient on both sides of the boundary must be the same, and in this case zero, then the total flow in the region of finite conductivity must be its normal component.

FLOW FROM POINT-SOURCE AND LINE-SOURCE

When dealing with the flow field only and discussing tubes of flow, we established in equation (118) the fact that the total volume-discharge along any given stream tube was constant, so that, as the tube contracted, the velocity of flow, or specific discharge **q**, must proportionately increase. Now, when we relate the rate of flow with the potential gradient, we obtain

$$\mathbf{q}A = -A\sigma \operatorname{grad} \Phi , \tag{144}$$

which tells us that, as any stream tube in a region of uniform conductivity contracts, the potential gradient correspondingly increases and equidifferent surfaces of constant potential crowd correspondingly closer together. Thus, in any region where the streamlines converge, the potential gradient will increase in the direction of convergence in proportion to the increase of **q**.

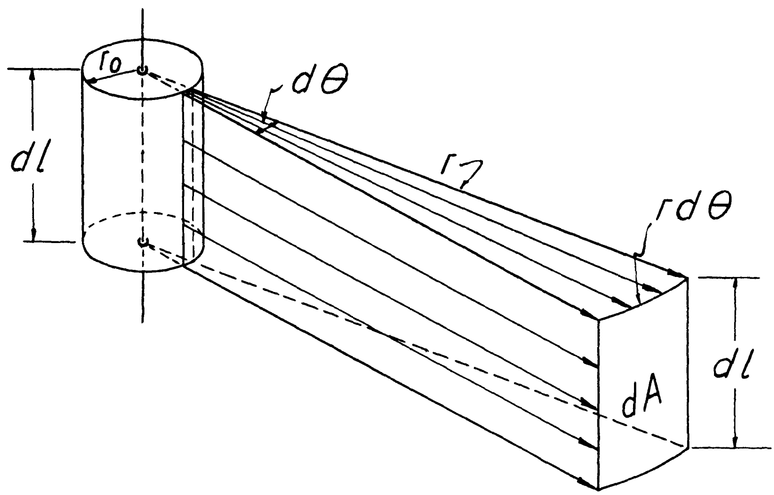

FIG. 22.—Elementary stream tube in flow from line- or cylindrical-source

Two cases of such convergence are of considerable importance because of their ease of calculation and because they are frequently approximated in practice. One of these is flow toward or away from a line-sink or line-source, where the flowlines are all normal to the line and radial with respect to it as a center. Radial flow into a well that extends from top to bottom of an aquifer with upper and lower impermeable boundaries is an example of this type of flow. Flow from a region of uniform permeability into a drainage ditch is, in some cases, a good approximation to it.

The other case is radial flow away from or toward a point-source or point-sink through media of uniform permeability.

For each of these two types of flow let us now determine both the magnitude of the specific discharge q and the potential Φ at all points in the field. For this we shall regard the discharge as positive when the flow is outward from a source and negative when directed inward toward a sink.

Consider the case when the flow is radially away from a line-source. Let this be represented by a cylinder of indefinite length and of radius r_0. Let the potential on the surface of the cylinder be Φ_0. Consider next the volume discharge dQ through a flow tube subtending upon the cylinder a length dl and a circumferential angle $d\theta$. At the distance r_0 the cross-sectional area of this tube will be

$$dA_0 = r_0 \cdot d\theta \cdot dl\,, \tag{145}$$

and at distance r this will become

$$dA = r \cdot d\theta \cdot dl\,. \tag{146}$$

Since the discharge over all cross sections of the tube is constant, then

$$dQ = q_0 r_0 \cdot d\theta \cdot dl = qr \cdot d\theta \cdot dl\,,$$

or

$$q = q_0 \cdot \frac{r_0}{r}\,, \tag{147}$$

which indicates that the rate of flow decreases inversely as the first power of the distance.

The potential is obtained by introducing Darcy's law into equation (147)

$$q = q_0 r_0 \cdot \frac{1}{r} = -\sigma \cdot \frac{d\Phi}{dr}\,, \tag{148}$$

from which

$$d\Phi = -\frac{q_0 r_0}{\sigma} \cdot \frac{dr}{r}$$

and

$$\Phi = \Phi_0 - \frac{q_0 r_0}{\sigma} \int_{r_0}^{r} \frac{dr}{r} = \Phi_0 - \frac{q_0 r_0}{\sigma} \cdot \log_e \frac{r}{r_0} . \tag{149}$$

In case the flow is inward toward a sink, q_0 is negative, so that the last term of equation (149) becomes positive.

Over a finite angle θ and length l the total discharge Q is

$$Q = q_0 r_0 \theta l \; ; \tag{150}$$

and when the flow is from all sides, θ is 2π radians, giving

$$Q = 2\pi q_0 r_0 l \, , \qquad \text{or} \qquad q_0 r_0 = \frac{Q}{2\pi l} . \tag{151}$$

Replacing $q_0 r_0$ in equation (149) with $Q/2\pi l$ then gives

$$\Phi = \Phi_0 - \frac{Q}{2\pi\sigma l} \cdot \log_e \frac{r}{r_0} , \tag{152}$$

showing that for this case the potential decreases or increases, depending upon whether the flow is outward or inward, logarithmically with distance.

Next consider radial flow from a point-source which we will take as the center of an equipotential sphere of radius r_0 and potential Φ_0. Let $d\Omega$ be an elementary solid angle whose area of cross section at distance r_0 is dA_0, and at distance r is dA. Through this the discharge is

$$dQ = q_0 \cdot dA_0 = q \cdot dA = q_0 r_0^2 \cdot d\Omega = q r^2 \cdot d\Omega . \tag{153}$$

Hence,

$$q = q_0 r_0^2 \cdot \frac{1}{r^2} , \tag{154}$$

showing that in this case the intensity of the flow varies with the inverse square of the distance. By Darcy's law

$$q = -\sigma \cdot \frac{d\Phi}{dr} = q_0 r_0^2 \cdot \frac{1}{r^2}, \tag{155}$$

from which

$$d\Phi = -\frac{q_0 r_0^2}{\sigma} \cdot \frac{dr}{r^2}$$

and

$$\Phi = \Phi_0 - \frac{q_0 r_0^2}{\sigma} \int_{r_0}^{r} \frac{dr}{r^2} = \Phi_0 + \frac{q_0 r_0^2}{\sigma} \left(\frac{1}{r} - \frac{1}{r_0} \right). \tag{156}$$

For inward flow the sign of q_0 becomes negative, changing the sign of the last term.

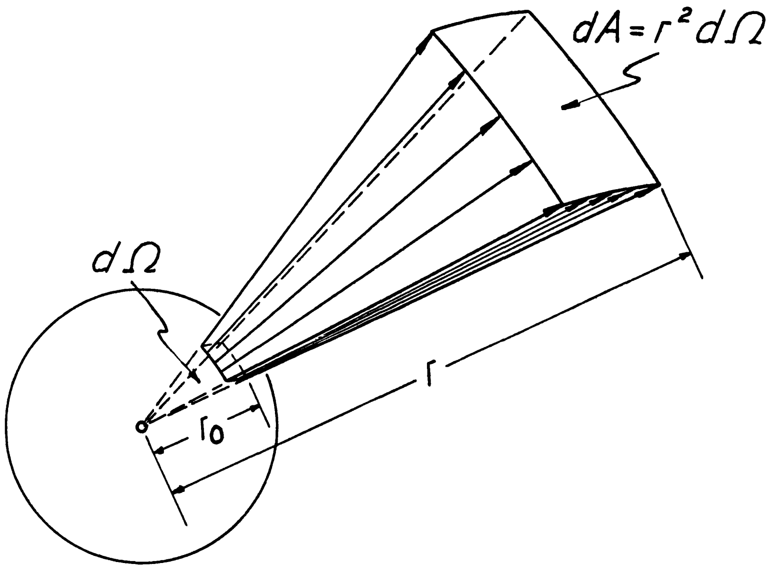

FIG. 23.—Elementary stream tube in flow from point- or spherical-source

For flow of this kind the flow commonly occurs over either half or the whole of the space about the point. For half-space the total discharge is

$$Q = 2\pi r_0^2 q_0, \tag{157}$$

and for whole space

$$Q = 4\pi r_0^2 q_0 \, , \tag{158}$$

from which

$$q_0 \, r_0^2 = \frac{Q}{2\pi} \quad \text{or} \quad \frac{Q}{4\pi} \, , \tag{159}$$

depending upon which of the two cases is dealt with. Taking the first, which corresponds to flow from or toward a point upon an impermeable plane boundary of a medium of uniform permeability and great extent, and introducing this into equation (156), we obtain

$$\Phi = \Phi_0 + \frac{Q}{2\pi\sigma}\left(\frac{1}{r} - \frac{1}{r_0}\right) . \tag{160}$$

In either of these two cases, let Φ_1 be the potential at a point P_1 of radial distance r_1, and Φ_2 that of a point P_2 of distance r_2. Then for flow toward a line-sink by equation (152)

$$\left.\begin{aligned} \Phi_1 &= \Phi_0 - \frac{Q}{2\pi\sigma l} \log_e \frac{r_1}{r_0} \, , \\ \Phi_2 &= \Phi_0 - \frac{Q}{2\pi\sigma l} \log_e \frac{r_2}{r_0} \, , \end{aligned}\right\} \tag{161}$$

and

$$\Phi_2 - \Phi_1 = \frac{Q}{2\pi\sigma l} \log_e \frac{r_1}{r_2} = g(h_2 - h_1) \, , \tag{162}$$

where h_2 and h_1 are the manometer heights at points P_2 and P_1.

Solving equation (162) for σ then gives

$$\sigma = \frac{Q}{2\pi l g(h_2 - h_1)} \log_e \frac{r_1}{r_2} = \frac{k\rho}{\eta} \, , \tag{163}$$

where k is the permeability of the medium. The heights h_2 and h_1 may be the elevations of water in two observation wells, while Q is the steady discharge of a pumping well.

For radial flow through half-space toward a point we have from equation (160)

$$\left.\begin{aligned}\Phi_1 &= \Phi_0 + \frac{Q}{2\pi\sigma}\left(\frac{1}{r_1} - \frac{1}{r_0}\right), \\ \Phi_2 &= \Phi_0 + \frac{Q}{2\pi\sigma}\left(\frac{1}{r_2} - \frac{1}{r_0}\right),\end{aligned}\right\} \tag{164}$$

and

$$\Phi_2 - \Phi_1 = \frac{Q}{2\pi\sigma}\left(\frac{1}{r_2} - \frac{1}{r_1}\right) = g(h_2 - h_1) . \tag{165}$$

Solving this for σ gives

$$\sigma = \frac{Q}{2\pi g(h_2 - h_1)}\left(\frac{1}{r_2} - \frac{1}{r_1}\right) = \frac{k\rho}{\eta} . \tag{166}$$

Equations (163) and (166) afford a means of determining permeability in the field when flow systems approximating those for which the two equations were developed exist. Thiem[10] appears to have been the first to utilize this principle in measuring permeability, and we may speak of it as the "Thiem method," although the method of analysis here employed differs markedly from the one used by him, and only one of the cases here used, that of flow to a line-sink, was considered by him.

The principle is by no means confined to the cases discussed; it is applicable to any flow system for which it is possible to deduce theoretically the potential field and the flow field from the known boundary conditions. The above cases were only chosen as an illustration.

Wenzel[11] has recently made use of the Thiem method for permeability measurements in the United States.

Another important property of the flow field derives from the principle of superposition. For example, suppose we have two point-sources at points O_1 and O_2 distance a apart inside a medium of uniform permeability whose extent in all directions is large, com-

[10] G. Thiem, *Hydrologische Methoden* (Leipzig, 1906).

[11] Leland K. Wenzel, "The Thiem Method for Determining Permeability of Water-bearing Materials," *U.S.G.S. Water-Supply Paper 697A* (1936).

pared with a. Let the discharge from one of these be Q_1 and from the other Q_2. Let r_1 and r_2 be the bi-polar co-ordinates of any point P in space measured from O_1 and O_2. If only the source Q_1 operates, then all flowlines will extend radially from O_1, and at $P(r_1, r_2)$ the flow vector will have the magnitude

$$q_1 = \frac{Q_1}{4\pi r_1^2} \tag{167}$$

and a direction parallel to r_1.

If only the source Q_2 operates, all flowlines will radiate from O_2 and at $P(r_1, r_2)$ the magnitude of the flow vector will be

$$q_2 = \frac{Q_2}{4\pi r_2^2}\,, \tag{168}$$

and its direction will be parallel to r_2.

If both Q_1 and Q_2 operate simultaneously, then $\mathbf{q}_1$ and $\mathbf{q}_2$ will be the two components of the flow, and the resultant flow will be the vector sum

$$\mathbf{q} = \mathbf{q}_1 + \mathbf{q}_2\,. \tag{169}$$

To obtain the resultant flowlines throughout the field by this method we should have to repeat the operations of equations (167),

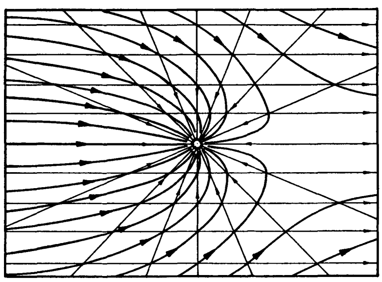

FIG. 24.—Graphical method of compounding two plane flow fields. The component fields are radial flow toward a line sink and rectilinear flow in the same plane. The flowlines are so spaced that between each pair the same discharge occurs. The method is *not applicable to other than plane fields.*

(168), and (169) for a large number of points in space before we could sketch them in—a very difficult and tedious process.

Suppose, however, we consider the potentials at the point $P(r_1, r_2)$ due to each source operating alone, and then the two together. For Q_1 alone

$$\Phi_1 = \Phi_{10} + \frac{Q_1}{4\pi\sigma}\left(\frac{1}{r_1} - \frac{1}{r_{10}}\right), \tag{170}$$

and for Q_2 alone

$$\Phi_2 = \Phi_{20} + \frac{Q_2}{4\pi\sigma}\left(\frac{1}{r_2} - \frac{1}{r_{20}}\right). \tag{171}$$

Then for both sources operating simultaneously we should have

$$\Phi = \Phi_1 + \Phi_2 . \tag{172}$$

But in this case the potentials are scalar quantities and so add algebraically, giving us

$$\Phi = \Phi_{10} + \Phi_{20} + \frac{Q_1}{4\pi\sigma}\left(\frac{1}{r_1} - \frac{1}{r_{10}}\right) + \frac{Q_2}{4\pi\sigma}\left(\frac{1}{r_2} - \frac{1}{r_{20}}\right), \tag{173}$$

by means of which we obtain the potential for every point $P(r_1, r_2)$ in space. The flowlines are then everywhere normal to the surfaces $\Phi = \text{constant}$.

In the case that Q_1 is a source and Q_2 an equal sink, with $\Phi_{20} = -\Phi_{10}$ and $r_{20} = r_{10}$, equation (182) reduces to

$$\Phi = \frac{Q}{4\pi\sigma}\left(\frac{1}{r_1} - \frac{1}{r_2}\right), \tag{174}$$

the gradient of which defines a flow field identical with the lines of force about an idealized bar magnet or with the lines of flow of an electric current between point electrodes through a space of uniform electrical conductivity.

By a generalization of this process the resultant potential at each point is always the sum of the partial potentials due to separate causes:

$$\Phi = \Phi_1 + \Phi_2 + \Phi_3 + \cdots + \Phi_n , \tag{175}$$

and always

$$\mathbf{q} = -\sigma \operatorname{grad} \Phi . \tag{176}$$

We have already demonstrated that upon all impermeable boundaries of a region of flow equipotential surfaces terminate perpendicularly. It remains to be seen how else such surfaces may terminate. For example, can an equipotential surface close completely upon itself, enclosing a volume of the flow region? Can an equipotential surface terminate inside a flow region away from all boundaries? Can two equipotential surfaces intersect each other or coincide?

In dealing with these problems let us keep in mind that we are assuming the presence of only a single homogeneous fluid of sensibly constant density, so that in the field of flow all voids are occupied by the fluid. The boundaries of the field of flow may be either permeable or impermeable. The flow field thus defined constitutes a closed volume through which the flow must satisfy equation (116), which states that the net outward flow across its boundaries must be zero. Consequently, in such a region all flow tubes must terminate upon the permeable boundaries, one terminus upon a fluid entrance or source, and the other upon a fluid exit or sink. Within the region at all points the flow is solenoidal, that is, devoid of absolute sources or sinks.

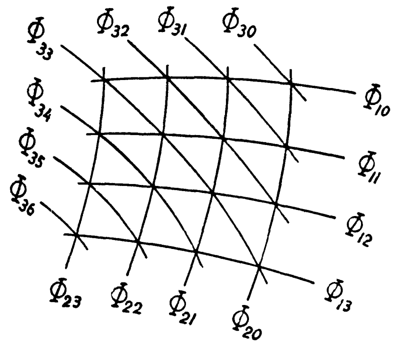

FIG. 25.—Graphical addition to potential fields. Φ_3 is the sum of Φ_1 and Φ_2.

One of the fundamental properties of an equipotential surface is that it separates a region of higher from one of lower potential. Consequently, across any given surface the flow always takes place in the same direction, that is, a single flowline never crosses the same equipotential surface twice. If an equipotential surface were to close completely upon itself, then we should have a volume into or out of which all flowlines converged or diverged. But this could only be possible provided the enclosed volume were a region in which the fluid was annihilated or created—a region containing either absolute sinks or sources. Since the existence of these has already been ruled out, then we may say that in the flow of an incompressible fluid it is impossible for an equipotential surface to terminate upon itself and completely enclose any given volume of space.

The same is true of the steady flow of a compressible fluid, only in this case the continuity equations apply to the mass discharge vector **j** instead of the volume discharge **q**, so that a completely closed equipotential surface for the steady motion of any kind of a fluid would involve a violation of the principle of the conservation of matter.

What we have here called "sinks" and "sources" inside a region of flow are actually achieved by a suitable arrangement of impermeable boundaries. We get something approaching a point sink inside a flow region by terminating a tube at that point. What we have actually done is to extend the impermeable boundaries in the form of a hollow cylinder to the given point. If the flow is from the medium to the end of the tube, then all near-by equipotential surfaces close completely about this point, except for the area of cross section of the tube. But this is the area of fluid exit, and the potential inside the tube is markedly different from that immediately outside. Consequently, even in this case, the surfaces cannot close but must terminate perpendicularly upon the impermeable boundaries of the tube.

Can equipotential surfaces terminate inside the flow region away from all boundaries? Equipotential surfaces must exist at all points for which there is a potential gradient and, hence, for which the fluid is in motion. If an equipotential surface should terminate away from all boundaries, this would imply that we had gone from a region in which the fluid was in motion into one where the fluid was completely stationary and, hence, at constant potential. If such an equipotential region existed adjacent to a region where the fluid was in motion, then we should have different equipotential surfaces terminating upon this stagnant region. This would enable us to carry a unit mass of fluid from a point upon an equipotential surface Φ_1 along this surface into the stagnant region and thence to a surface of higher potential Φ_2 with zero work. Then, carrying it directly back to its initial point would yield the positive work $\Phi_2 - \Phi_1$. In this manner we should achieve a perpetual-motion mechanism and violate the principle of the conservation of energy. Hence we must conclude that no equipotential surface can terminate in this manner but must continue to a boundary. A corollary to this is that in a region occupied by a homogeneous fluid it is impossible to have one part of the fluid in motion and another part stagnant, except when the two are separated by impermeable boundaries—a proposition of some significance with regard to certain theories of ore deposition.

Can two equipotential surfaces intersect or coincide? Let one of these have the value Φ and the other $\Phi + \Delta\Phi$. Let the distance between them along the flowline be Δn. Then by Darcy's law

$$\mathbf{q} = \underset{\Delta n \to 0}{\text{limit}} \left(-\sigma \frac{\Delta\Phi}{\Delta n} \right) . \qquad (177)$$

Since $\Delta\Phi$ is the constant difference of potential between the two surfaces, then as they approached each other, Δn would approach zero, and $\Delta\Phi/\Delta n$ would become infinite, corresponding to an infinite value of $\mathbf{q}$. Hence, at all points where the flow vector $\mathbf{q}$ is finite, no two different equipotential surfaces can ever coincide or intersect.

Exceptions to this occur at certain *singular points* in the flow region, one example of which is the flow around a sharp exterior angle of an impermeable boundary. As the corner is approached, the different equipotential surfaces which are normal to each of the two faces approach a point of coincidence. In the immediate neighborhood of such a point the velocity increases markedly, and Darcy's law loses its validity.

THE BOUNDARIES BETWEEN INHOMOGENEOUS FLUIDS

So far we have dealt only with the flow of a single homogeneous fluid through a region all voids of which were supposedly filled completely by that fluid. By our usage a homogeneous fluid is one whose density is a function of pressure only. Continuing this usage, we now wish to investigate the bounding surface between two different homogeneous fluids occupying different parts of a single flow region. Different homogeneous fluids may be immiscible, like water and oil, or miscible in all proportions, like water and alcohol. Where they are immiscible, their interface is a discrete surface of separation; where they are miscible, no such precise surface exists, but its place is taken by a zone of diffusion.

For immiscible fluids with their discrete surfaces of separation there exists an amount of surface energy where the two fluids are in mutual contact, and also other surface energies between each of the fluids separately and the solid medium. These give rise to capillary forces which act as a modifying influence upon an equilibrium that would obtain were such forces nonexistent. In many instances the effect of these modifying influences of capillarity are not of great importance; in other instances, especially in dealing with the oil-water relations of petroleum production, their effect appears to be far from negligible.

In ground-water problems the surface energies are effective along the air-water contacts but are lacking along the fresh-water—salt-water contacts. For simplicity we shall first assume that capillary forces do not exist, and then later introduce their effects as a modifying influence upon the results thus obtained.

While dealing with a single fluid we have had to consider only a single potential. In dealing with several fluids we shall have as many different potentials as there are fluids. These we shall define in such a manner that the definitions will be valid at all points in space not occupied by an impermeable solid. Thus, at each point in space, regardless of which fluid occupies that space, there will be a particular value of each of the several potentials. Also, there will be a complete scalar field in each potential throughout the region of space considered. In addition to this, each fluid will occupy a particular domain of its own with a surface of contact, or interface, between adjoining regions occupied by different fluids.

To deal with this more complex system we must generalize our notation somewhat if we are to avoid ambiguity and confusion. We must be able to distinguish the potentials of the several different fluids and the particular values assigned to each. In addition to this we must distinguish the separate regions occupied by the different fluids. The first two requirements are met by the use of two subscripts, the first identifying the fluid concerned, and the second, the particular value assigned. Thus Φ_{ij}, where i and j are 1, 2, 3, ... , signifies the jth particular value of the potential of the ith fluid. Where no particular value is intended, the second subscript will be omitted. Frequently it will be necessary to show that a given operation is to be performed in a region occupied by a specified fluid. This will be indicated by $(\ )_k$, where the parentheses enclose the operation and $k = 1, 2, 3, \ldots$ indicates the fluid in whose region it is to be performed. Thus $(\text{grad } \Phi_2)_1$ indicates the gradient of the potential of fluid 2 in the region occupied by fluid 1.

The potential of the ith fluid at any given point will be

$$\Phi_i = U + \int_{p_0}^{p} \frac{dp}{\rho_i}, \tag{178}$$

where ρ_i is its density. For liquids this simplifies to

$$\Phi_i = U + \frac{p - p_0}{\rho_i}. \tag{179}$$

At and near the earth's surface, gravity is sufficiently nearly constant that we may set $U = gz$. Then

$$\Phi_i = gz + \frac{p - p_0}{\rho_i}. \tag{180}$$

Also,

$$\frac{\partial \Phi_i}{\partial s} = g \cdot \frac{\partial z}{\partial s} + \frac{1}{\rho_i} \cdot \frac{\partial p}{\partial s} \tag{181}$$

and

$$-\text{grad } \Phi_i = +\mathbf{g} - \frac{1}{\rho_i} \text{grad } p \tag{182}$$

This last equation is a vector equation, and each of its terms is a force per unit of mass that would act upon an element of the ith fluid placed at the specified point. It is important to observe that, while at a given point both $\mathbf{g}$ and $-\text{grad } p$ are vectors with fixed magnitudes and directions, the vector $-1/\rho_i \text{ grad } p$ has a fixed direction but a variable magnitude, depending upon the fluid density ρ_i. Consequently, for elements of fluids of different densities placed at the same point, the driving forces per unit of mass, $-\text{grad } \Phi_i$, will always be of different magnitudes and, in general, in different directions. Similarly, for different fluids, the equipotential surfaces $\Phi_i = \text{constant}$, passing through a given point, will in general not be parallel to one another, since for each fluid the equipotential surfaces are normal to grad Φ_i.

The importance of this is that it provides a separation mechanism for fluids of different densities. For instance, if we began initially with an emulsion of two immiscible fluids of different densities, the particles of each would be driven at different rates and, in general, along nonparallel paths. This process would continue until the two fluids were segregated into different regions.

Here, again, we are neglecting surface energies, so that for this to be strictly true it is necessary that the fluid particles be large enough for the driving forces to predominate over forces due to surface tensions and colloidal phenomena.

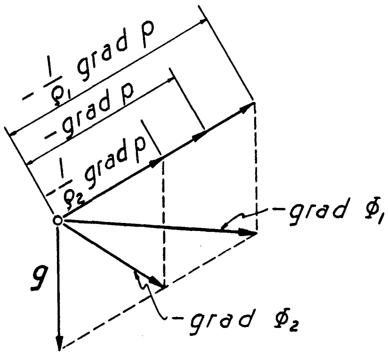

FIG. 26.—Forces acting upon elements of fluids of different densities placed at the same point.

It is not meant to imply that an existing fluid segregation must have arisen in the above manner, but rather to point out that such a mechanism exists. Suppose, for example, that we have two segregated fluids with a common interface. For this arrangement to be a stable one it is necessary for an element of either fluid, if placed in the region occupied by the other, to be repelled and driven back into its own territory. The mechanism for this is that described above.

Now, if we consider a region in different parts of which different fluids are segregated, there will be as many different fluid interfaces as there are pairs of different fluids in contact with one another. It is our purpose to study these interfaces; but since relationships valid for one should be valid for all, we need only consider a single interface between the two fluids of densities ρ_1 and ρ_2, both in steady motion.

For a given steady state of motion a fluid interface will be fixed in space and will exhibit many of the properties of a solid, impermeable boundary. If the segregation of the two fluids is complete, neither fluid can cross the boundary, so that the normal components of the flow of each must be zero and the flowline on each side tangential to the boundary, although the flowlines of the two fluids on the opposite sides of the boundary may make any angle in that plane with each other.

In case the segregation is not complete, the normal components of flow will not be zero; and in each region where fluid is crossing the boundary, the flowlines will form some angle, $\gamma < 90°$, with the surface normal directed into that region. An important instance of this sort will be considered later.

Considering, for the present, the case of complete fluid segregation, then the interface for each state of flow will act like a rigid impermeable boundary, but for each change in the state of flow of either of the fluids the interface will adjust itself to a new position of equilibrium. It represents, therefore, a configuration of dynamic equilibrium. Let us see how this is related to the potentials of the two fluids.

Let n_1 be the normal of the interface into the region occupied by the fluid of density ρ_1, and n_2 that into the region of the fluid of density ρ_2. Assuming that our fluids are liquids, then at any point in either region

$$\left.\begin{aligned}\Phi_1 &= gz + \frac{p - p_0}{\rho_1}\,,\\ \Phi_2 &= gz + \frac{p - p_0}{\rho_2}\,.\end{aligned}\right\} \tag{183}$$

Since stability requires that an element of either fluid in the domain of the other be driven back to its own territory, this implies corresponding repulsive force components along the surface normals in the respective regions. These can only be the result of an increase of the potential of each fluid in the domain of the other in the direction of its surface normal. Thus, for stability,

$$\left.\begin{aligned}\left(\frac{\partial \Phi_2}{\partial n_1}\right)_1 &> 0\,,\\ \left(\frac{\partial \Phi_1}{\partial n_2}\right)_2 &> 0\,.\end{aligned}\right\} \tag{184}$$

These are obtained in a convenient form if we eliminate $p - p_0$ from equation (183) and solve for Φ_1 and Φ_2, respectively; we then get

$$\left.\begin{aligned} \Phi_1 &= -\frac{\rho_2 - \rho_1}{\rho_1} \cdot gz + \frac{\rho_2}{\rho_1} \cdot \Phi_2 \,, \\ \Phi_2 &= +\frac{\rho_2 - \rho_1}{\rho_2} \cdot gz + \frac{\rho_1}{\rho_2} \cdot \Phi_1 \,. \end{aligned}\right\} \tag{185}$$

Then

$$\left.\begin{aligned} \left(\frac{\partial \Phi_2}{\partial n_1}\right)_1 &= \frac{\rho_2 - \rho_1}{\rho_2} \cdot g \cdot \frac{\partial z}{\partial n_1} + \frac{\rho_1}{\rho_2}\left(\frac{\partial \Phi_1}{\partial n_1}\right)_1 , \\ \left(\frac{\partial \Phi_1}{\partial n_2}\right)_2 &= -\frac{\rho_2 - \rho_1}{\rho_1} \cdot g \cdot \frac{\partial z}{\partial n_2} + \frac{\rho_2}{\rho_1}\left(\frac{\partial \Phi_2}{\partial n_2}\right)_2 . \end{aligned}\right\} \tag{186}$$

But since the normal components of flow across the interface are zero,

$$\left(\frac{\partial \Phi_1}{\partial n_1}\right)_1 = \left(\frac{\partial \Phi_2}{\partial n_2}\right)_2 = 0 \,, \tag{187}$$

so that equations (186) simplify to

$$\left.\begin{aligned} \left(\frac{\partial \Phi_2}{\partial n_1}\right)_1 &= \frac{\rho_2 - \rho_1}{\rho_1} \cdot g \cdot \frac{\partial z}{\partial n_1} , \\ \left(\frac{\partial \Phi_1}{\partial n_2}\right)_2 &= -\frac{\rho_2 - \rho_1}{\rho_1} \cdot g \cdot \frac{\partial z}{\partial n_2} . \end{aligned}\right\} \tag{188}$$

Stability requires that both the right-hand terms be positive. Bearing in mind that n_2 is oppositely directed from n_1,

$$\frac{\partial z}{\partial n_2} = -\frac{\partial z}{\partial n_1} ,$$

so that if n_1 is upward-directed, $\partial z / \partial n_1$ is positive and $\partial z / \partial n_2$ is negative, and vice versa. Hence if n_1 is the upward-directed normal, stability requires that $\rho_1 < \rho_2$; if n_2 is upward-directed, ρ_2 must be less than ρ_1. In both cases the less dense liquid must occupy the upper space. When the interface is vertical,

$$\frac{\partial z}{\partial n_1} = \frac{\partial z}{\partial n_2} = 0\,,$$

and neutral equilibrium results.

SLOPE OF THE INTERFACE

Let us now investigate the slope of the interface as a function of the states of flow of the two fluids or of their two potentials in the fields of flow. Eliminating $p - p_0$ from equation (183) and solving for z gives us

$$z = \frac{1}{g}\left(\frac{\rho_2}{\rho_2 - \rho_1} \cdot \Phi_2 - \frac{\rho_1}{\rho_2 - \rho_1} \cdot \Phi_1\right). \tag{189}$$

This is the elevation of any point whatever in either of the two regions, or of any other region for that matter, for which the two potentials Φ_1 and Φ_2 are known. Or, if we like, we may think of every

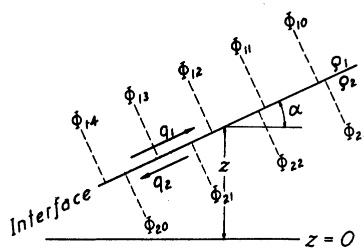

FIG. 27.—Tilt of fluid interface due to flow of both fluids

point in space being characterized by particular values of the three quantities z, Φ_1, and Φ_2, whose relations to each other are given by equation (189), so that if any two of them are known the third is uniquely determined.

Now let us consider the potentials and the components of flow in a vertical plane section through the system. Let s be the trace of the interface along this section. Equation (189) is valid for points on the interface as well as elsewhere; and while in general we do not know the values of Φ_1 and Φ_2 at given points on the interface, if α is its angle of slope measured positively upward in the direction s,

$$\sin \alpha = \frac{\partial z}{\partial s} = \frac{1}{g}\left(\frac{\rho_2}{\rho_2 - \rho_1} \cdot \frac{\partial \Phi_2}{\partial s} - \frac{\rho_1}{\rho_2 - \rho_1} \cdot \frac{\partial \Phi_1}{\partial s}\right). \qquad (190)$$

Then, since

$$\left.\begin{aligned} \frac{\partial \Phi_2}{\partial s} &= -\frac{q_{2s}}{\sigma}, \\ \frac{\partial \Phi_1}{\partial s} &= -\frac{q_{1s}}{\sigma}, \end{aligned}\right\} \qquad (191)$$

where q_{1s} is the component in the direction s of the flow vector $\mathbf{q}_1$ of the ρ_1-fluid in the section we are considering, and q_{2s} that of the ρ_2-fluid, these can be substituted into equation (190), transforming the latter into

$$\sin \alpha = \frac{\partial z}{\partial s} = \frac{1}{\sigma g}\left(-\frac{\rho_2}{\rho_2 - \rho_1} \cdot q_{2s} + \frac{\rho_1}{\rho_2 - \rho_1} \cdot q_{1s}\right). \qquad (192)$$

If both fluids are static $q_{1s} = q_{2s} = 0$ and $\sin \alpha = 0$, showing that for these conditions the equilibrium configuration for the fluid interface is a horizontal surface. Starting from a static state, let us keep the ρ_2-fluid static and increase q_{1s} by successive steps. Then q_{2s} will be zero, and equation (192) will become

$$\sin \alpha = \frac{\partial z}{\partial s} = \frac{1}{\sigma g} \cdot \frac{\rho_1}{\rho_2 - \rho_1} \cdot q_{1s} = -\frac{1}{g} \cdot \frac{\rho_1}{\rho_2 - \rho_1} \cdot \frac{\partial \Phi_1}{\partial s}. \qquad (193)$$

Hence, if $\rho_1 < \rho_2$, as the rate of flow q_{1s} of the less dense ρ_1-fluid is increased, $\sin \alpha$ will also increase, and the interface will tilt upward in the direction of the flow.

Now, if we repeat this procedure, keeping $q_{1s} = 0$, and gradually increasing q_{2s}, we shall have

$$\sin \alpha = \frac{\partial z}{\partial s} = -\frac{1}{\sigma g} \cdot \frac{\rho_2}{\rho_2 - \rho_1} \cdot q_{2s} = +\frac{1}{g} \cdot \frac{\rho_2}{\rho_2 - \rho_1} \cdot \frac{\partial \Phi_2}{\partial s}, \qquad (194)$$

showing that for flow in the direction s of the denser ρ_2-fluid the interface will tilt downward in the direction of the flow.

If both fluids flow simultaneously, the angle α will be the resultant of the two tilts, as determined by equation (192). If they both flow in the same direction, the two tilts will oppose each other, and α will be intermediate between its values, owing to the two flows occurring separately, becoming zero when

$$\sin \alpha = 0 = \frac{1}{\sigma g}\left(-\frac{\rho_2}{\rho_2 - \rho_1} \cdot q_{2s} + \frac{\rho_1}{\rho_2 - \rho_1} \cdot q_{1s}\right)$$

or when

$$\frac{q_{1s}}{q_{2s}} = \frac{\dfrac{\partial \Phi_1}{\partial s}}{\dfrac{\partial \Phi_2}{\partial s}} = \frac{\rho_2}{\rho_1}\,. \tag{195}$$

When the two fluids flow in opposite directions, the two component tilts augment one another, and the resultant is greater than that due to either alone.

For simplicity, let us now confine our attention to the equations involving the potentials or their gradients, bearing in mind that we can always switch to the rates of flow by means of the equation of transformation (191) when we wish to do so.

We note, according to the general equation (190) and the particular ones (193) and (194), that $\sin \alpha$ increases with the appropriate increase of the components of the potential gradients along the interface in the section considered. Since the maximum value of $\sin \alpha$ is unity, it will be interesting to see what happens when this value is reached. Taking the general equation (190), we see that

$$(\sin \alpha)_{\max} = 1 = \frac{1}{g}\left(\frac{\rho_2}{\rho_2 - \rho_1} \cdot \frac{\partial \Phi_2}{\partial s} - \frac{\rho_1}{\rho_2 - \rho_1} \cdot \frac{\partial \Phi_1}{\partial s}\right)_{\max}, \tag{196}$$

which tells us that the maximum value the difference between the two terms in the parentheses can have is g. When only one of the fluids is flowing at a time, the potential gradient of the other will be zero, and

$$\left.\begin{aligned} (\sin \alpha)_{\max} &= +1 = -\frac{1}{g} \cdot \frac{\rho_1}{\rho_2 - \rho_1}\left(\frac{\partial \Phi_1}{\partial s}\right)_{\max}, \\ (\sin \alpha)_{\max} &= +1 = +\frac{1}{g} \cdot \frac{\rho_2}{\rho_2 - \rho_1}\left(\frac{\partial \Phi_2}{\partial s}\right)_{\max}, \end{aligned}\right\}$$

or

$$\left.\begin{aligned}\left(\frac{\partial\Phi_1}{\partial s}\right)_{\max} &= -g\cdot\frac{\rho_2-\rho_1}{\rho_1}\,,\\ \left(\frac{\partial\Phi_2}{\partial s}\right)_{\max} &= +g\cdot\frac{\rho_2-\rho_1}{\rho_2}\,,\end{aligned}\right\} \tag{197}$$

when the ρ_1- and the ρ_2-fluids, respectively, are flowing.

Equations (196) and (197) are statements that, when each fluid is flowing alone, the component along the interface of its potential gradient cannot exceed a certain critical finite value and that, when both flow simultaneously, the difference of the components of their potential gradients cannot exceed a specified critical finite amount. What is the meaning of this physically? At first thought it seems to imply that these critical potential gradients are the maximum obtainable. That this interpretation is absurd is evident when we consider that the potential gradients are our independent variables and are subject to being given any values we choose over a very wide range.

What it does signify is this: That for values of the gradients greater than those stated, the interfacial surface ceases to exist. When only a single fluid is flowing, this comes about either by lateral migration until the interface coincides with an impermeable boundary or by the confluence of two stream tubes approaching from different directions. In both cases the static fluid is squeezed out from the region where the critical value of the potential gradient is exceeded.

When both fluids are flowing, lateral migration of the vertical interface to an impermeable boundary is impossible, because this would reduce the cross section of the stream tubes of one of the fluids to zero. In this case the interface migrates laterally; and, as it does so, it increases the stream-tube area of one fluid and decreases that of the other, reducing the first potential gradient and increasing the second. This migration will continue until the critical maximum difference of the two potential gradients is again established.

In fact, the same kind of thing happens where only one of the fluids is flowing. If the critical potential gradient is exceeded, the interface shifts laterally in such a direction as to increase the area of the stream tubes and decrease the potential gradient. This will continue either until the potential gradient is reduced to its critical value or until an impermeable boundary, or another set of stream tubes of the same fluid, is encountered.

THE ELEVATION OF THE FLUID INTERFACE

The elevation of any point at which the potentials of the two fluids is known is given by equation (189). In particular, this is true of a point upon the fluid interface. The difficulty is that, when both

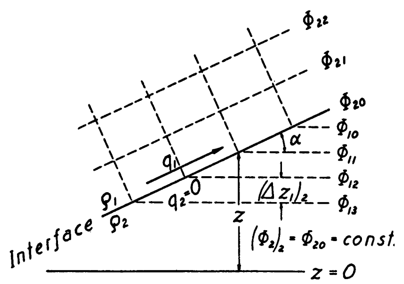

FIG. 28.—Tilt of interface and equipotential surfaces of both fluids when one fluid flows while the other is static.

fluids are flowing simultaneously, we do not know the values of the two potentials at points on the interface; nor is there any convenient method of determining them in most instances. Consequently, to obtain the equation of the interface for this general state of flow we should be obliged to rely upon the integration of slope equations like (190), which in all but the simplest cases is difficult to do.

Fortunately, in practice the most important problems encountered are those in which only one of the two fluids is flowing, while the other remains sensibly static and, hence, at constant potential. The interface then becomes an equipotential surface in terms of this potential, and equation (189) becomes a linear equation between the values of the remaining potential along the interface and the elevation of its points.

For example, let the ρ_1-fluid flow while the ρ_2-fluid remains stationary. Then in the ρ_2-region the potential Φ_2 will be constant, say of the value Φ_{20}. Then throughout the ρ_2-region and upon the interface the elevation of any given surface Φ_1 = constant will be

$$z = \frac{1}{g}\left(\frac{\rho_2}{\rho_2 - \rho_1} \cdot \Phi_{20} - \frac{\rho_1}{\rho_2 - \rho_1} \cdot \Phi_1\right). \tag{198}$$

Since, when Φ_1 = constant, z = constant also, it is clear that in the ρ_2-region the family of surfaces Φ_1 = constant are horizontal. When they intersect the interface, they then refract into perpendicularity with that surface in the ρ_1-region.

Also from equation (198) the family of equidifferent surfaces Φ_1 = constant are equidistantly spaced. If $\Delta\Phi_1$ be the potential difference between successive surfaces, their spacing is

$$(\Delta z_1)_2 = -\frac{1}{g} \cdot \frac{\rho_1}{\rho_2 - \rho_1} \cdot \Delta\Phi_1 . \tag{199}$$

In a similar manner, if we let the ρ_1-fluid remain stationary while the ρ_2-fluid flows, and let Φ_{10} be the constant value of the potential (Φ_1), the elevation along the interface and inside the ρ_1-region of a surface Φ_2 = constant will be

$$z = \frac{1}{g}\left(\frac{\rho_2}{\rho_2 - \rho_1} \cdot \Phi_2 - \frac{\rho_1}{\rho_2 - \rho_1} \cdot \Phi_{10}\right). \tag{200}$$

These surfaces will likewise be horizontal in the ρ_1-region with a spacing for a potential difference $\Delta\Phi_2$ of

$$(\Delta z_2)_1 = \frac{1}{g} \cdot \frac{\rho_2}{\rho_2 - \rho_1} \cdot \Delta\Phi_2 . \tag{201}$$

In these two cases the potential gradients of the moving fluid in the region occupied by the static one will be vertical. In the separate ones of these two cases, by solving equations (198) and (200) for Φ_1 and Φ_2, respectively, we obtain

$$\left.\begin{aligned} (\Phi_1)_2 &= -\frac{\rho_2 - \rho_1}{\rho_1} \cdot gz + \frac{\rho_2}{\rho_1} \cdot \Phi_{20} , \\ (\Phi_2)_1 &= +\frac{\rho_2 - \rho_1}{\rho_2} \cdot gz + \frac{\rho_1}{\rho_2} \cdot \Phi_{10} , \end{aligned}\right\} \tag{202}$$

from which

$$\left.\begin{aligned}\left(\frac{\partial \Phi_1}{\partial z}\right)_2 &= -\frac{\rho_2 - \rho_1}{\rho_1} \cdot g\,, \\ \left(\frac{\partial \Phi_2}{\partial z}\right)_1 &= +\frac{\rho_2 - \rho_1}{\rho_2} \cdot g\,.\end{aligned}\right\} \qquad (203)$$

Letting $\rho_2 > \rho_1$, then, when the ρ_2-region is at constant potential, the Φ_1 potential decreases upward or increases downward at the rate $[(\rho_2 - \rho_1)/\rho_1]g$. In the ρ_1-region, when this is at constant potential the potential Φ_2 increases upward at the rate $[(\rho_2 - \rho_1)/\rho_2]g$. Stated vectorially,

$$\left.\begin{aligned}(\text{grad } \Phi_1)_2 &= \frac{\rho_2 - \rho_1}{\rho_1} \cdot \mathbf{g}\,, \\ (\text{grad } \Phi_2)_1 &= -\frac{\rho_2 - \rho_1}{\rho_2} \cdot \mathbf{g}\,,\end{aligned}\right\} \qquad (204)$$

where **g** is now the downward-directed vector of the acceleration due to gravity.

THREE-FLUID CASE

Now let us consider a special case involving three fluids. Let ρ_1, ρ_2, and ρ_3 be their three densities, and let $\rho_1 < \rho_2 < \rho_3$. Let the ρ_2-fluid flow while the ρ_1 and the ρ_3 fluids remain static. We shall now have the fluids arranged in three layers according to their densities, with the least dense uppermost, with a 1, 2 and a 2, 3 interface.

The three potentials will be

$$\left.\begin{aligned}\Phi_1 &= gz + \frac{p - p_0}{\rho_1}\,, \\ \Phi_2 &= gz + \frac{p - p_0}{\rho_2}\,, \\ \Phi_3 &= gz + \frac{p - p_0}{\rho_3}\,.\end{aligned}\right\} \qquad (205)$$

Besides, we shall have

$$\left.\begin{aligned}(\Phi_1)_1 &= \text{constant} = \Phi_{10}\,, \\ (\Phi_2)_2 &\neq \text{constant} = \Phi_2\,, \\ (\Phi_3)_3 &= \text{constant} = \Phi_{30}\,.\end{aligned}\right\} \qquad (206)$$

Eliminating $p - p_0$ from contiguous pairs of equations (205) and solving for z, and then substituting the values from equations (206) into the results, gives for the elevations of points on the two interfaces

$$\left.\begin{aligned} z_{12} &= \frac{1}{g}\left(\frac{\rho_2}{\rho_2 - \rho_1}\cdot\Phi_2 - \frac{\rho_1}{\rho_2 - \rho_1}\cdot\Phi_{10}\right), \\ z_{23} &= \frac{1}{g}\left(\frac{\rho_3}{\rho_3 - \rho_2}\cdot\Phi_{30} - \frac{\rho_2}{\rho_3 - \rho_2}\cdot\Phi_2\right). \end{aligned}\right\} \tag{207}$$

The heights z_{12} and z_{23} of these equations are the elevations above the standard datum of the intersection of the two respective interfaces with a surface Φ_2 = constant. The difference between these two elevations is

$$\left.\begin{aligned} z_{12} - z_{23} = \frac{1}{g}\Bigg[\left(\frac{\rho_2}{\rho_2 - \rho_1} + \frac{\rho_2}{\rho_3 - \rho_2}\right)\Phi_2 & \\ - \frac{\rho_1}{\rho_2 - \rho_1}\cdot\Phi_{10} - \frac{\rho_3}{\rho_3 - \rho_2}\cdot\Phi_{30}\Bigg]. & \end{aligned}\right\} \tag{208}$$

These two elevations are, however, not of points or lines one vertically above the other.

Also, equidifferent surfaces Φ_2 = constant refract across the two interfaces into the two static regions as equally spaced horizontal surfaces, the scales of the spacing being different in the two cases. For a potential difference $\Delta\Phi_2$ the two spacings are

$$\left.\begin{aligned} (\Delta z_2)_1 &= \frac{1}{g}\cdot\frac{\rho_2}{\rho_2 - \rho_1}\cdot\Delta\Phi_2, \\ (\Delta z_2)_3 &= -\frac{1}{g}\cdot\frac{\rho_2}{\rho_3 - \rho_2}\cdot\Delta\Phi_2, \end{aligned}\right\} \tag{209}$$

and their ratios are given by

$$\frac{(\Delta z_2)_1}{(\Delta z_2)_3} = -\frac{\rho_3 - \rho_2}{\rho_2 - \rho_1}. \tag{210}$$

In the most important three-fluid case in ground-water problems the fluids are the air, fresh water, and salt water. While the foregoing equations have been derived for liquids, in this case the air is at sensibly constant potential, and so we can deal with it with no alteration of the above equations. We let ρ_1 be the density of the air, ρ_2 that of fresh water, and ρ_3 that of salt water. Then by choosing our standard datum at sea-level and 1 atmosphere as our standard pressure,

$$\left.\begin{aligned} (\Phi_1)_1 &= gz_1 + \int_{p_0}^{p_1} \frac{dp}{\rho_1} = 0\,, \\ \Phi_2 &= gz_2 + \frac{p_2 - p_0}{\rho_2} = gh\,, \\ (\Phi_3)_3 &= gz_3 + \frac{p_3 - p_0}{\rho_3} = 0\,. \end{aligned}\right\} \tag{211}$$

Also, the density of the air is negligible compared with that of water, and so may be neglected. In this case equations (207) simplify to

$$\left.\begin{aligned} z_{12} &= \frac{1}{g} \cdot \frac{\rho_2}{\rho_2 - \rho_1} \cdot \Phi_2 = h\,, \\ z_{23} &= -\frac{1}{g} \cdot \frac{\rho_2}{\rho_3 - \rho_2} \cdot \Phi_2 = -\frac{\rho_2}{\rho_3 - \rho_2} \cdot h\,. \end{aligned}\right\} \tag{212}$$

The difference between these elevations for a given surface $\Phi_2 =$ constant is then

$$z_{12} - z_{23} = \left(1 + \frac{\rho_2}{\rho_3 - \rho_2}\right) h = \frac{\rho_3}{\rho_3 - \rho_2} \cdot h\,. \tag{213}$$

The spacings of the fresh-water equipotential surfaces in regions occupied by air and salt water, respectively, are obtained from equation (209) and are

$$\left.\begin{aligned} (\Delta z_2)_1 &= \frac{1}{g} \cdot \frac{\rho_2}{\rho_2 - \rho_1} \cdot \Delta\Phi_2 = \Delta h\,, \\ (\Delta z_2)_3 &= -\frac{1}{g} \cdot \frac{\rho_2}{\rho_3 - \rho_2} \cdot \Delta\Phi_2 = -\frac{\rho_2}{\rho_3 - \rho_2} \cdot \Delta h\,, \end{aligned}\right\} \tag{214}$$

and their ratio is

$$\frac{(\Delta z_2)_1}{(\Delta z_2)_3} = -\frac{\rho_3 - \rho_2}{\rho_2}\,. \tag{215}$$

It should also be added that, in accordance with an earlier stipulation, we have here neglected the surface tension of the fresh-water–air interface. Strictly speaking, the elevation z_{12} is that of the water table. The actual water-air interface inside a permeable medium will be somewhat higher than this.

In all of these equations h refers to the elevation of the free surface in a manometer tube filled with fresh water and terminated at the point whose potential is to be determined.

Again it should be emphasized that the elevations z_{12} and z_{23} are the elevations above the standard datum of the intersections with the upper and lower interfaces of a particular fresh-water equipotential surface, and only in the case that all three fluids are in static equilibrium do these elevations occur one vertically above the other. In all other cases the surface Φ_2 = constant will be curved and the vertical distance between the upper and lower interfaces will be greater—sometimes but not always by a negligible amount—than the difference $z_{12} - z_{23}$ as given by equation (213).

Where a given surface Φ_2 = constant intersects the upper and lower interfaces, the sines of the angles of slope of these surfaces are

$$\left.\begin{aligned} \sin \alpha_{12} &= \frac{1}{g} \cdot \frac{\rho_2}{\rho_2 - \rho_1} \cdot \frac{\partial \Phi_2}{\partial s_{12}}, \\ \sin \alpha_{23} &= -\frac{1}{g} \cdot \frac{\rho_2}{\rho_3 - \rho_2} \cdot \frac{\partial \Phi_2}{\partial s_{23}}. \end{aligned}\right\} \qquad (216)$$

If it be assumed that $\partial\Phi_2/\partial s_{12} = \partial\Phi_2/\partial s_{23}$, which in many cases is a good approximation, then the ratios of the sines of these two angles is

$$\frac{\sin \alpha_{12}}{\sin \alpha_{23}} = -\frac{\rho_3 - \rho_2}{\rho_2 - \rho_1} = -\frac{\rho_3 - \rho_2}{\rho_2}, \qquad (217)$$

which is the same as the ratio of the equipotential surface spacings given by equation (215). For fresh water and sea water the ratio $(\rho_3 - \rho_2)/\rho_2 \cong \frac{1}{40}$.

FLOW ACROSS AN INTERFACE

In the foregoing we have assumed that no flow occurred across the fluid interfaces. There are several important instances for which this is not true. If the segregation is not complete, discrete elements of finite size of either fluid in the territory of the other will be repelled and driven back into their own territory. The fall of raindrops from the atmosphere toward the air-water interface is an important instance of this kind; the rise of gas bubbles through a liquid is another. For processes in the reverse direction we have diffusion and evaporation.

In either case, when a given region is gaining or losing its own fluid across a fluid interface, the flowlines and flow tubes will no longer be parallel to the interface but will terminate upon it. The interface will accordingly act as a source or a sink. Likewise the equipotential surfaces will now approach the interface at some angle different from 90°. The configuration of the interface, however, will still be determined by the two sets of potentials of the contiguous fluids in the same manner as that we have already discussed.

REFRACTION OF A FLUID INTERFACE ACROSS A PLANE BOUNDARY BETWEEN MEDIA OF DIFFERENT PERMEABILITIES

Let us now investigate the behavior of the interface between two fluids when this intersects the plane boundary between media of different permeabilities. These two surfaces will now divide our field of flow into regions a and b of different permeabilities and into regions 1 and 2 occupied by different fluids. We let ρ_1 be the density of one of the fluids and ρ_2 that of the other, with $\rho_1 < \rho_2$, and let k_a and k_b be the two permeabilities.

For simplicity we consider only the components of the flow and the corresponding potential gradients in a single arbitrary vertical plane, and we allow only one of the fluids to flow while the other remains static and at constant potential. The moving fluid is taken to be a liquid, though, if the denser fluid is flowing, the less dense one may be a gas. The moving fluid is made always to cross the a–b plane in the direction from a to b.

We let β be the angle of slope of the a–b plane in the section considered measured upward from the horizontal in the direction of the flow, s_{ab} the length along its trace, and n its normal. Let α_a and α_b be the limiting angles of slope of the fluid interface in the regions a and b

in the vertical section considered as the a–b plane is approached, and s_{12} the length along its trace in the direction of the flow.

Now let the ρ_1-fluid flow while the ρ_2-fluid remains stationary. Then the surfaces Φ_1 = constant will approach the fluid interface perpendicularly in the ρ_1-region and then refract into a family of

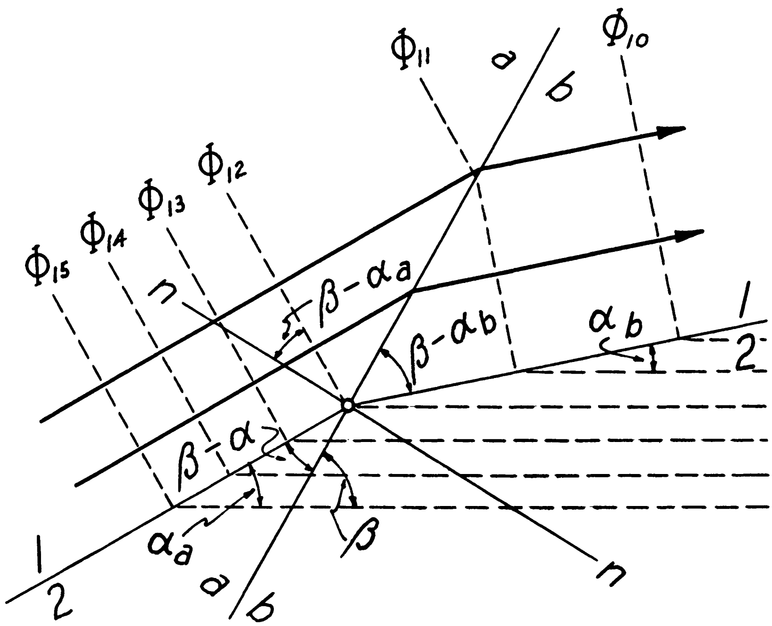

FIG. 29.—Diagram for the identification of quantities employed in discussing intersection of fluid interface with boundary between regions of different permeabilities.

equally spaced horizontal surfaces in the ρ_2-region. Inside the ρ_1-region along the a–b plane as the fluid interface is approached, the boundary conditions which must be satisfied are the following: The normal components of flow across the a–b plane must be the same in both the regions a and b, and the tangential components of the potential Φ_1 must be the same in both these regions:

$$\left(\frac{\partial \Phi_1}{\partial s_{ab}}\right)_{1a} = \left(\frac{\partial \Phi_1}{\partial s_{ab}}\right)_{1b}, \qquad (218)$$

$$(q_{1n})_{1a} = (q_{1n})_{1b}. \qquad (219)$$

Each of these quantities can now be evaluated separately in terms of the gradient of Φ_1 in region 2, which is fixed, and of the permeabilities k_a and k_b and the angles α_a, α_b, and β. We then get

$$\left.\begin{aligned}\left(\frac{\partial\Phi_1}{\partial s_{ab}}\right)_{1a} &= \left(\frac{\partial\Phi_1}{\partial s_{12}}\right)_{1a} \cos(\beta-\alpha_a) = \left(\frac{\partial\Phi_1}{\partial z}\right)_2 \sin\alpha_a\cdot\cos(\beta-\alpha_a)\,,\\ \left(\frac{\partial\Phi_1}{\partial s_{ab}}\right)_{1b} &= \left(\frac{\partial\Phi_1}{\partial s_{12}}\right)_{1b} \cos(\beta-\alpha_b) = \left(\frac{\partial\Phi_1}{\partial z}\right)_2 \sin\alpha_b\cdot\cos(\beta-\alpha_b)\,.\end{aligned}\right\} \quad (220)$$

Similarly,

$$\left.\begin{aligned}-(q_{1n})_{1a} &= \sigma_a\left(\frac{\partial\Phi_1}{\partial n}\right)_{1a} = \sigma_a\left(\frac{\partial\Phi_1}{\partial s_{12}}\right)_{1a}\sin(\beta-\alpha_a)\\ &\qquad = \sigma_a\left(\frac{\partial\Phi_1}{\partial z}\right)_2 \sin\alpha_a\cdot\sin(\beta-\alpha_a)\,,\\ -(q_{1n})_{1b} &= \sigma_b\left(\frac{\partial\Phi_1}{\partial n}\right)_{1b} = \sigma_b\left(\frac{\partial\Phi_1}{\partial s_{12}}\right)_{1b}\sin(\beta-\alpha_b)\\ &\qquad = \sigma_b\left(\frac{\partial\Phi_1}{\partial z}\right)_2 \sin\alpha_b\cdot\sin(\beta-\alpha_b)\,.\end{aligned}\right\} \quad (221)$$

Equating the separate equations (220) gives

$$\sin\alpha_a\cdot\cos(\beta-\alpha_a) = \sin\alpha_b\cdot\cos(\beta-\alpha_b)\,. \qquad (222)$$

Equations (221) give

$$\sigma_a\sin\alpha_a\cdot\sin(\beta-\alpha_a) = \sigma_b\sin\alpha_b\cdot\sin(\beta-\alpha_b)\,. \qquad (223)$$

But, since

$$\sigma_a = \frac{k_a\rho_1}{\eta_1}\,, \qquad \text{and} \qquad \sigma_b = \frac{k_b\rho_1}{\eta_1}\,,$$

equation (223) reduces to

$$k_a\sin\alpha_a\cdot\sin(\beta-\alpha_a) = k_b\sin\alpha_b\cdot\sin(\beta-\alpha_b)\,. \qquad (224)$$

Equations (222) and (224) give the relations between α_a, α_b, β, k_a, and k_b, which must simultaneously be satisfied at the intersection of the fluid interface with the a–b plane. Since the fluid crosses this plane from a to b,

$$\alpha_a \gtreqless \beta\,, \qquad \alpha_b \gtreqless \beta\,.$$

The angles α_a and α_b are the dependent variables whose values are to be determined when the values of the independent variables β, k_a, and k_b are given.

Equation (222) is given a simple geometrical interpretation if we draw a circle whose vertical upward-directed diameter is taken as a unit vector. Then at the lower end of this diameter we draw a horizontal line. Taking the intersection of these two lines as a polar origin, we lay off the angles α_a, α_b, and β measured from the hori-

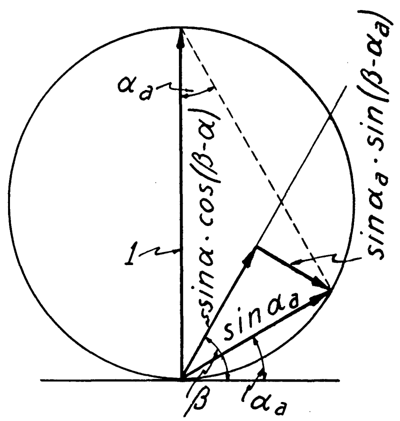

FIG. 30.—Graphical representation of quantities in equations (222) and (224)

zontal. The lines through the origin making the angles α_a, α_b, and β, respectively, form chords to the circle. The length of the α_a-chord is numerically to sin α_a, and its projection upon the β-chord is equal to $\sin \alpha_a \cdot \cos (\beta - \alpha_a)$; the length of the α_b-chord is equal to $\sin \alpha_b$, and its projection upon the β-chord is $\sin \alpha_b \cdot \cos (\beta - \alpha_b)$.

When this has been done, it is seen by inspection that equation (222) can only be satisfied by

$$\alpha_a = \alpha_b \, . \tag{225}$$

Introducing this result into equation (224) transforms the latter into

$$k_a \sin \alpha_a \cdot \sin (\beta - \alpha_a) = k_b \sin \alpha_a \cdot \sin (\beta - \alpha_a) \, . \tag{226}$$

Before solving this for α_a, however, another condition needs to be taken into account, namely, that, owing to instability of the interface which would result for values greater than this, the maximum possible value of α_a is 90°. We must therefore make a distinction between the solutions of equation (226) for values of β less than 90° and values greater than 90°. For $\beta < 90°$ the solutions of equation (226) for different values of k_a and k_b are

$$k_a \neq k_b: \qquad \sin \alpha_a \cdot \sin (\beta - \alpha_a) = 0 , \qquad \alpha_a = 0 \text{ or } \beta . \quad (227)$$

$$k_a = k_b: \quad 0 \lesseqgtr \sin \alpha_a \cdot \sin (\beta - \alpha_a) \lesseqgtr 1 , \qquad 0 \lesseqgtr \alpha_a \lesseqgtr 90° . \quad (228)$$

When $\alpha_a = 0$, both fluids are in static equilibrium so that their interface is a horizontal surface upon which the *a–b* plane exerts no influence. When the ρ_1-fluid flows, however, α_a cannot be zero; and if $k_a \neq k_b$, then the interface must twist around so as to approach and leave the *a–b* plane tangentially. Physically, this is somewhat similar to an easement curve on railroads, where the centrifugal force is made to increase gradually from zero to a maximum and then decline gradually to zero again. In the fluid case discontinuities are avoided when the interface approaches and leaves the *a–b* plane tangentially. When $k_a = k_b$ the problem reduces to that of the flow through a homogeneous medium.

When $90° < \beta < 180°$, $0 \lesseqgtr \alpha_a \lesseqgtr 90°$; and, strictly speaking, it is impossible to satisfy equation (226) except for $\alpha_a = 0$, corresponding to zero flow, or for $k_a = k_b$, corresponding to a homogeneous medium.

Evidently in this case, when $k_a \neq k_b$, some discontinuity has to take place, and our problem resolves itself into determining the nature of the discontinuity. When $k_b > k_a$, this is fairly obvious. Imagine the flow to occur from *a* to *b* while the angle β is slowly increased from a value less than 90° to one greater than this amount. Until β reaches 90° and the *a–b* plane is vertical, the angles α will equal β, and the interface will approach the surface tangentially. When it is rotated farther, α will remain at 90°. The discharge q_{1a} will then be at its maximum in region *a*, but across the *a–b* plane in region *b* there will be the same potential gradient as in *a* but a higher permeability. This will produce a faster velocity of flow than in *a;* but since the total discharge cannot be different from that in *a*, the ρ_1-fluid in *b* will break up and will either mix with the ρ_2-fluid or, if they are immiscible, drops will form and rise vertically upward through a ρ_2-matrix until the interface in the b region is encountered and crossed.

Upgrade along the a–b plane the flowlines in the a region will gradually swing around toward the surface normal. When the angle θ_a they make with this normal reaches such a value that

$$\tan \theta_a = \frac{k_a}{k_b} \tan \theta_b = \frac{k_a}{k_b} \tan (180^\circ - \beta) , \qquad (229)$$

corresponding to the tangent law of refraction with the flowlines in the b region vertical, continuity of flow will again be established.

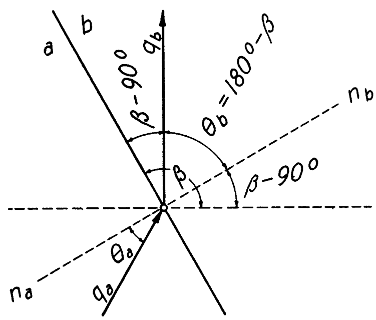

FIG. 31.—The angles referred to in equation (229)

Thus, for $\beta > 90^\circ$ and $k_b > k_a$ there will be a finite displacement of the interface in the b-region upward along the a–b plane from its terminus in the a-region, with both the angles α_a and α_b equal to 90°. Between the termini of these two interfaces the flow across the a–b plane will give rise in the b-region either to a zone of diffusion and mixing or to one in which the ρ_1-fluid will percolate upward through the ρ_2-fluid until it again crosses the interface.

When $90^\circ < \beta < 180^\circ$, and $k_a > k_b$, and $0 < \alpha_a \lesseqqgtr 90^\circ$, the problem is indeterminate, for no value of α_a within this range will satisfy equation (226). Neither can there be a fluid discontinuity across the a–b plane, because the higher velocity fluid is now in the a-region; nor is a displacement of the fluid interface in the b-region *down* the a–b plane possible because this would correspond to a flow in the direction of increasing potential. What must happen is that a zone of distortion exists in the immediate vicinity of the intersection of the two surfaces so that equation (226) does not represent the true state of affairs. Evidently the interface is continuous across the a–b plane; and in the special cases that the flow is zero or that $k_a = k_b$, $\alpha_a = \alpha_b$. More than this one cannot say.

If we keep the ρ_1-fluid stationary and let the ρ_2-fluid flow, equations analogous to those from (218) to (229) may be derived. In this case the angle of tilt α of the interface will be downward in the direction of the flow, and the angle β should be measured in the same direction as α.

The most important case of this in practice occurs when k_a is finite and k_b infinite, corresponding to the flow from an underground region into an open space. In this case, when $0 < \beta < 90°$, $\alpha_a = \beta$ and the interface will approach the a–b plane tangentially. Since, however, discharge across this plane must occur (by hypothesis), the flowlines in the a-region away from the interface must swing around so as to intersect it. Across the boundary in the b-region either this fluid is absorbed by mixing and diffusion or else it forms a thin sheet and flows upward along the a–b plane until it encounters a horizontal interface between static bodies of the two fluids.

When $90° < \beta < 180°$, a similar thing occurs except that $\alpha_a = 90°$, and either the ρ_1-fluid mixes or diffuses into the ρ_2-fluid or else it rises vertically by percolation until it crosses the horizontal interface between static bodies of the two fluids.

In both of these cases the zone along the a–b plane between the terminus upon it of the interface in the a-region and the static fluid interface in the b-region is called a *surface of seepage*. Along the surface of seepage the ρ_1-fluid occupies one side of the a–b plane; and, except for a possible veneer of the ρ_2-fluid on the b-side, the ρ_2-fluid occupies the other. Consequently, along the a–b plane in the zone of seepage the component of the gradient of Φ_1 is

$$\frac{\partial \Phi_1}{\partial s_{ab}} = (\text{grad } \Phi_1)_{1a} \sin \theta = (\text{grad } \Phi_1)_2 \sin \beta \,,$$

or

$$(\text{grad } \Phi_1)_{1a} = (\text{grad } \Phi_1)_2 \frac{\sin \beta}{\sin \theta} \,, \tag{230}$$

where θ is the angle these flowlines make with the normal to the a–b plane.

For constant values of β, $(\text{grad } \Phi_1)_2 \sin \beta$ is constant, indicating that along the surface of seepage the tangential component of the potential gradient is constant, so that, as the values of θ decrease from 90° to 0°, the corresponding value of $(\text{grad } \Phi_1)_{1a}$ increases from zero to infinity.

In the a-region at the fluid interface the flowlines approach the a–b plane either tangentially or vertically, and here the respective

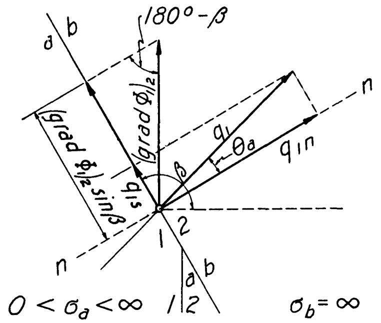

FIG. 32.—Manner of increase of the rate of flow of fluid 1 from region of finite permeability into one of infinite permeability occupied by fluid 2, as θ_a decreases.

values of θ are 90° or $180° - \beta$. Then beyond the interface between the static bodies of ρ_1 and ρ_2 fluids in the b-region, the tangential component of the gradient of Φ_1 abruptly becomes zero, because the ρ_1-body is a region of Φ_1 = constant. Opposite to this the flowlines in the a-region must approach the a–b plane perpendicularly, so that here the angle θ is zero.

Hence, in the a-region, from one edge of the zone of seepage to the other, the direction of the flowlines with respect to the plane-normal must change from $\theta = 0$ to $\theta = 180° - \beta$, or 90°. This change cannot be discontinuous with distance because, if it were, we should have two separate flowlines in the a-region intersecting upon the a–b plane, or a finite stream tube converging to zero cross section. Consequently,

$$\frac{\partial s_{ab}}{\partial \theta} > 0 . \tag{231}$$

If we let A and B mark the edges of the zone of seepage with A corresponding to $\theta = 0$ and B to $\theta =$ maximum, then the width of the zone of seepage is

$$s_{ab}\Big|_A^B = \int_{\theta_A}^{\theta_B} \frac{\partial s_{ab}}{\partial \theta} \cdot d\theta > 0 \,. \tag{232}$$

Consequently, so long as $\theta_A \neq \theta_B$, the width of the zone of seepage must always be greater than zero.[12]

The normal component of the flow across the zone of seepage will be

$$\left.\begin{aligned} q_{1n} &= -\sigma_a\,(\text{grad }\Phi_1)_{1a}\cos\theta = -\sigma_a\,(\text{grad }\Phi_1)_2\,\frac{\sin\beta}{\sin\theta}\cdot\cos\theta \\ &= -\sigma_a\,(\text{grad }\Phi_1)_2\,\sin\beta\cdot\cot\theta \,. \end{aligned}\right\} \tag{233}$$

For a given case all the quantities in equation (233) are constant except q_{1n} and θ, so that, as θ decreases from its maximum value to zero, along the zone of seepage, q_{1n} simultaneously increases along this zone from a minimum (zero if $\theta_{\max} = 90^\circ$) to a maximum approaching infinity as the static fluid interface is approached. Actually, in this vicinity the velocities exceed those for which Darcy's law is valid, so that all this really means is that q_{1n} reaches a finite maximum value at that point.

APPLICATION TO GROUND WATER

In ground-water problems where our fluids are the air, fresh water, and salt water, we have an interface between the fluids air–fresh-water and air–salt-water and between fresh-water–salt-water. The last is only stable underground, being broken up in open basins by diffusion and mixing due to motions resulting from very small potential gradients. Consequently, in open basins we may have a more or less uniform body of salt water or of fresh water, but not a static layering of both with a discrete fluid interface.

When fresh water flows into a fresh-water basin having an air-water interface, a zone of seepage of finite width will form along the ground surface above the level of the water in the basin. If this

[12] Both Dachler and Muskat have discussed this problem for the air–fresh-water interface and have reached conclusions in conformity with those here.

ground slope is less than vertical, the air-water interface will approach it tangentially; if it overhangs, the air-water interface will approach vertically. In the zone of seepage the water will either flow or drip to the basin of fresh water or else will evaporate into the air.

If the basin of water is salt water with a static air–salt-water interface and the fresh water is flowing, then both the air–fresh-water and the air–salt-water interfaces will be effective. Above the water surface in the open basin there will be a fresh-water–air zone of seepage, and below that level a fresh-water–salt-water zone of seepage. In

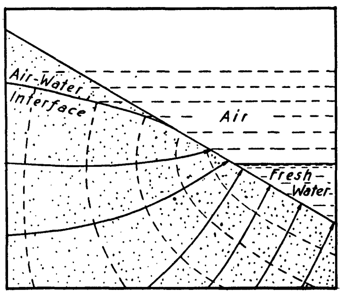

FIG. 33

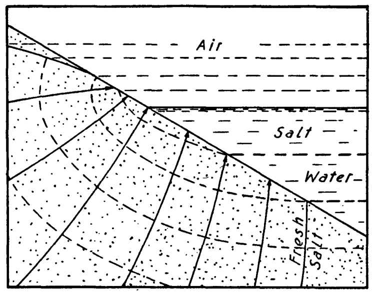

FIG. 34

FIG. 33.—Approximate manner of flow of fresh water into open space containing bodies of air and fresh water.

FIG. 34.—Approximate manner of flow of fresh water into open space containing air and salt water. Note orientation of the flowlines and the two interfaces as boundary to open space is approached.

the first zone the fresh water will either evaporate or will flow on the ground surface to the salt-water basin; in the second it will flow into it directly. In both cases mixing of the fresh and salt water will occur, and the fresh water, as such, will disappear.

For the fresh-water–air zone of seepage the angle β will, in most cases, be less than 90°, while for the fresh-water–salt-water zone it will almost always be greater than 90°. Hence, under a basin of salt water the fresh-water–salt-water interface will approach the basin boundary vertically.

Also, for the flow into a salt-water basin the entire fresh-water discharge must flow across the combined zones of seepage, since the outer boundaries of these are the two fluid interfaces. Consequently, their combined width cannot be zero but must be finite and great enough to accommodate the discharge.

Since there is no equipotential body of fresh water in the basin, then along the surface of the ground (the *a–b* plane) the tangential components of the potential gradient will be

$$\left.\begin{aligned} \left(\frac{\partial \Phi_2}{\partial s_{ab}}\right)_1 &= (\text{grad } \Phi_2)_1 \sin \beta \,, \\ \left(\frac{\partial \Phi_2}{\partial s_{ab}}\right)_3 &= (\text{grad } \Phi_2)_3 \sin \beta \,, \end{aligned}\right\} \tag{234}$$

where the subscripts 1, 2, and 3 refer to air, fresh water, and salt water, respectively, and where s_{ab} is the distance along the upward slope of the *a–b* plane, whose angle of slope in both cases is β. The quantity $(\text{grad } \Phi_2)_1$ is a vector directed vertically upward, and $(\text{grad } \Phi_2)_3$ another directed vertically downward. Consequently, the components of these two gradients along the *a–b* plane (the surface of the ground) will be oppositely directed, each away from the salt-water–air interface. The intersection of the static salt-water–air surface with the surface of the ground will accordingly be a line across which the tangential component of the potential gradient along the *a–b* plane suddenly changes direction. The flowlines at the same time will have to make a transition from one side of the surface-normal to the other. In order for the angle θ not to change discontinuously with distance, the potential gradient and the rate of flow $\mathbf{q}$ must become very great at the point of this transition. Consequently, the intersection of the static salt-water–air surface with the surface of the ground will be the region in which the flow of the fresh water from the ground will be the most concentrated.

THE EFFECT OF CAPILLARITY

In all our discussions of fluid interfaces we have assumed that capillary forces were nonexistent. In the case of an interface between miscible fluids like fresh water and salt water this assumption is valid; but for water-air, water-oil, and other similar interfaces it is not valid.

When two fluids are in contact with each other and with the plane face of a solid, there are three surface tensions: T_{12}, T_{23}, and T_{31}, one between each pair of substances indicated by the subscripts 1, 2, and 3. These tensions are forces per unit of length along the surface. Let substance 3 be the solid, and let γ be the angle between the inter-

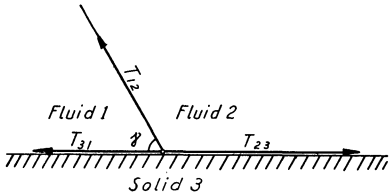

FIG. 35.—Surface tensions between two fluids and a plane solid

faces of fluid 1 with the solid and fluid 2. For equilibrium the components of all tensions parallel to the face of the solid must balance. Hence,

$$T_{31} + T_{12} \cos \gamma = T_{23} ,$$

or

$$T_{12} \cos \gamma = T_{23} - T_{31} . \tag{235}$$

These tensions are constants for given pairs of materials. When $\gamma < 90^\circ$, the fluid 1 is said to *wet* the solid; when $\gamma > 90^\circ$ (mercury on glass, for example), no wetting occurs. In the first instance, the line of junction of the three substances tends to move away from the region of fluid 1; in the second case, toward it.

When

$$T_{23} - T_{31} > T_{12} ,$$

the angle γ is zero, and fluid 1 is said to wet the solid completely. This is exhibited by a water-air interface on glass, quartz, and other rock-forming minerals.

Parallel to the face of the solid and directed from fluid 1 to fluid 2, there acts upon the interface 1, 2 a net force of $T_{12} \cos \gamma$ per unit of length of contact with the solid. This causes the interface to migrate in the direction of the force until the two fluids build up a difference of pressure, the force due to which counterbalances that due to the surface tension.

When dealing with two fluids occupying the pore spaces of a permeable solid, the interfaces whose surface tensions are to be dealt with are those of the two fluids and the solid framework. The walls of the solid, instead of being plane, are now rounded or angular. If the solid medium is macroscopically isotropic, the surfaces of the solid along the trace of the fluid interface will be randomly oriented. Microscopically, therefore, the fluid interface will be very complicated, having radii of curvature of the order of magnitude of the width of the open spaces in the rock. Macroscopically, however,

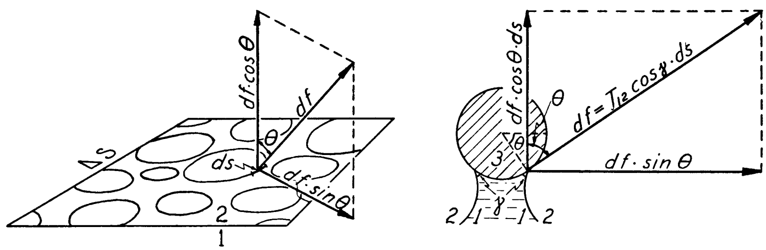

FIG. 36.—Macroscopic surface element of interface in sand between two fluids having surface tensions. *a*, Forces acting upon length *ds* of line of contact; *b*, detail of the same.

these minute irregularities are lost sight of and the fluid interface becomes smooth, with radii of curvature so large that macroscopic surface elements may generally be regarded as plane.

Now, upon the interface let us take a macroscopic surface element ΔS to which we erect a surface-normal in the direction from fluid 1 to fluid 2. Within this surface area we let ds be a microscopic element of length of the line of contact of the 1, 2 interface with the solid surface of the medium, and $d\mathbf{f}$ the element of capillary force normal to ds and parallel to the solid face. Let θ be the angle between the direction of $d\mathbf{f}$ and the surface-normal $\boldsymbol{n}$. The magnitude of this force will be

$$df = T_{12} \cos \gamma \cdot ds \; ; \tag{236}$$

and, if we resolve it into components tangential and normal to the surface ΔS, these will be

$$df_t = T_{12} \cos \gamma \cdot ds \cdot \sin \theta \tag{237}$$

and

$$df_n = T_{12} \cos \gamma \cdot ds \cdot \cos \theta \, , \qquad (238)$$

respectively. The total force over the area ΔS is then obtained by integration.

When it is considered that the tangential components of force $d\mathbf{f}_t$ are randomly oriented in the surface element ΔS and so have radial symmetry about the normal n, it is clear that these, considered vectorially, must cancel each other, and

$$\int_{\Delta S} d\mathbf{f}_t = 0 \, . \qquad (239)$$

The normal components, on the contrary, must be of the same sign algebraically, since the direction of the force element $d\mathbf{f}$ which makes the angle θ with the normal n is essentially restricted to the half-space occupied by one fluid or the other. Consequently,

$$f_n = \int_{\Delta S} df_n = T_{12} \cos \gamma \int_{\Delta S} ds \cdot \cos \theta \, . \qquad (240)$$

An analytical evaluation of $\int_{\Delta S} ds \cdot \cos \theta$ involves making assumptions regarding the microscopic structure of the medium and fluid interface, which at best can only be fair approximations, so that, like the permeability k, this quantity is more readily and accurately obtainable by experiment. We can, however, transform it into a more convenient form by writing

$$\int_{\Delta S} ds \cdot \cos \theta \equiv s \, \overline{\cos \theta} \, , \qquad (241)$$

where s is the total length in the area ΔS of the contact of the 1, 2 interface with the surfaces of the solid medium, and $\overline{\cos \theta}$ a mean or effective value of $\cos \theta$ for the whole macroscopic area. The quantity $\overline{\cos \theta}$ depends only upon the internal shape of the medium and not upon its size scale, and so it should be constant for geometrically similar media.

Combining equations (241) and (240) then gives us

$$f_n = T_{12} \cos \gamma \cdot s \overline{\cos \theta} \,. \tag{242}$$

This force, which is distributed over the fluid interface, causes it to migrate in the direction of f_n until a difference of pressure in the fluid bodies on opposite sides of the interface great enough to counteract it is developed. The amount of this pressure difference, or *capillary* pressure, is given by

$$p_2 - p_1 = p_c = \frac{f_n}{\epsilon \cdot \Delta S} = \frac{T_{12} \cos \gamma \cdot s \overline{\cos \theta}}{\epsilon \cdot \Delta S} \,. \tag{243}$$

For interfaces between the same two fluids in two geometrically similar media, a and b, of the same material

$$\frac{(p_c)_b}{(p_c)_a} = \frac{\left(\dfrac{T_{12} \cos \gamma \cdot s \overline{\cos \theta}}{\epsilon \cdot \Delta S}\right)_b}{\left(\dfrac{T_{12} \cos \gamma \cdot s \overline{\cos \theta}}{\epsilon \cdot \Delta S}\right)_a} = \frac{s_b}{s_a} \cdot \frac{\Delta S_a}{\Delta S_b} \,. \tag{244}$$

But

$$\frac{s_b}{s_a} = \frac{d_b}{d_a} \,, \qquad \text{and} \qquad \frac{\Delta S_a}{\Delta S_b} = \frac{d_a^2}{d_b^2} \,, \tag{245}$$

so that when these are introduced into equation (244) the latter reduces to

$$\frac{(p_c)_b}{(p_c)_a} = \frac{d_a}{d_b} \,, \tag{246}$$

where d_a and d_b are the mean grain diameters or other characteristic lengths of the two media. In general, for the same materials and geometrically similar media,

$$p_c = C \cdot \frac{1}{d} \,, \tag{247}$$

where C is a constant of proportionality.

In ground-water problems, across the fresh-water–air interface in a quartz sand the water wets the sand, and the angle γ is sensibly zero. Consequently, the force f_n on the interface is directed from the region occupied by the water to that occupied by the air.

Equilibrium is established when the pressure against the air exceeds that against the water on opposite sides of the interface by the amount p_c. The order of magnitude of this capillary pressure for different degrees of coarseness of the sand can be obtained by noting that for an unconsolidated mixed quartz sand of mean grain diameter of about 0.02 cm. the vertical increase of the manometer height of otherwise static water is about -15 cm. In this case

$$\rho g h_c = C \cdot \frac{1}{d}, \qquad \text{or} \qquad C = \rho g h_c d \,, \tag{248}$$

where h_c is the increase in the manometer height of the water (in this case negative) due to the capillary pressure against the water (also negative). Supplying the numerical values above gives

$$p_c \cong -1.5 \times 10^4 \text{ dyne/cm}^2 \,,$$

$$C \cong -300 \text{ dyne/cm}^2 \,.$$

This value for C is not based upon precise sedimentary analysis and so is to be taken only as an order of magnitude. Using it in this capacity, however, we obtain an idea of the pressure difference p_c and the manometer rise h_c for the static water-air interface in similar sediments of increasing fineness, by solving equation (245) for h_c and p_c when the grain diameters are given. A few such values are plotted in Table 1, from which it is clear that for sediments of large permeabilities the drop in manometer height is unlikely to exceed 1 meter, whereas for sediments with manometer drops greater than a meter the permeabilities become vanishingly small.

INFLUENCE OF CAPILLARITY UPON A TWO-FLUID INTERFACE

In our earlier discussion of fluid interfaces we assumed that p_c was zero, so that at one and the same point the pressure against which either of two fluids would have to be injected was the same. Also, it was assumed that the pressures at contiguous points on opposite sides of the fluid interface were the same. We now see that, if the two fluids are immiscible, they will always be separated by a fluid interface on opposite sides of which the pressures will have the difference p_c, where p_c depends upon the nature of the fluids and the medium in the manner already indicated. This is equally true whether the interface be that between large segregated bodies of the two fluids or whether it be that surrounding a small macroscopic volume of one fluid injected into the region occupied by the other.

TABLE 1

INCREASE OF MANOMETER HEIGHT AND CAPILLARY PRESSURE OF THE WATER-AIR INTERFACE IN SIMILAR SEDIMENTS

Rock Type	Grain Diameter d (Cm.)	Increase of Manometer Height h_c (Cm.)	Capillary Pressure p_c (Dynes/Cm²)
Gravel	1	-3×10^{-1}	-3×10^{2}
	10^{-1}	-3	-3×10^{3}
Sand	10^{-2}	$-3\times10^{+1}$	-3×10^{4}
Silt	10^{-3}	-3×10^{2}	-3×10^{5}
Clay	10^{-4}	-3×10^{3}	-3×10^{6}

To deal with this circumstance it is only necessary to extend our definition of fluid potential so as to include the work against interfacial capillary pressure. The potential of any fluid in any region is then

$$\Phi = gz + \int_{p_0}^{p+p_c} \frac{dp}{\rho}, \tag{249}$$

where ρ is the density of the injected fluid, p the pressure of the fluid occupying the point in question, and p_c the capillary pressure *against* the injected fluid due to the interfacial tensions of the two fluids at the point of injection. For the same fluids and medium, with the medium homogeneous as regards texture and size scale, p_c may be treated as a constant; otherwise its variation as a function of position must be allowed for.

The potentials of two fluids at the same point are now

$$\left.\begin{aligned}\Phi_1 &= gz + \int_{p_0}^{p+p_{c1}} \frac{dp}{\rho_1}\,,\\ \Phi_2 &= gz + \int_{p_0}^{p+p_{c2}} \frac{dp}{\rho_2}\,,\end{aligned}\right\} \tag{250}$$

which for fluids of constant densities reduces to the simpler form

$$\left.\begin{aligned}\Phi_1 &= gz + \frac{p - p_0}{\rho_1} + \frac{p_{c1}}{\rho_1}\,,\\ \Phi_2 &= gz + \frac{p - p_0}{\rho_2} + \frac{p_{c2}}{\rho_2}\,.\end{aligned}\right\} \tag{251}$$

These equations are analogous to equation (183), and again we may eliminate $p - p_0$ and solve for z, obtaining

$$z = \frac{1}{g}\left(\frac{\rho_2}{\rho_2 - \rho_1}\cdot\Phi_2 - \frac{\rho_1}{\rho_2 - \rho_1}\cdot\Phi_1 - \frac{p_{c2} - p_{c1}}{\rho_2 - \rho_1}\right). \tag{252}$$

Also, if we like, we may observe that

$$\left.\begin{aligned}\frac{p_{c1}}{\rho_2 - \rho_1} &= \frac{\rho_1}{\rho_2 - \rho_1}\cdot\frac{p_{c1}}{\rho_1} = \frac{\rho_1}{\rho_2 - \rho_1}\cdot\Phi_{c1}\,,\\ \frac{p_{c2}}{\rho_2 - \rho_1} &= \frac{\rho_2}{\rho_2 - \rho_1}\cdot\frac{p_{c2}}{\rho_2} = \frac{\rho_2}{\rho_2 - \rho_1}\cdot\Phi_{c2}\,,\end{aligned}\right\} \tag{253}$$

where Φ_{c1} and Φ_{c2} are the parts of the two potentials due to capillarity. Inserting these expressions into equation (252) then gives

$$z = \frac{1}{g}\left[\frac{\rho_2}{\rho_2 - \rho_1}(\Phi_2 - \Phi_{c2}) - \frac{\rho_1}{\rho_2 - \rho_1}(\Phi_1 - \Phi_{c1})\right]. \tag{254}$$

In setting up equations (249) and (250) it was supposed that both fluids were to be injected at the same point in a region occupied by the same fluid. Now, if z represents the elevation of a point on the 1, 2 interface, then we must think of both fluids being injected on the same side of the existing interface against a pressure either of p_1 or p_2, of one fluid or the other.

If the injection is made on the side of fluid 1 against pressure p_1,

$$\left.\begin{aligned} \frac{p_{c1}}{\rho_1} &= \Phi_{c1} = 0\,, \\ \frac{p_{c2}}{\rho_2} &= \Phi_{c2} \neq 0\,. \end{aligned}\right\} \qquad (255)$$

Or, if the injection is made on the side of the interface occupied by fluid 2 against pressure p_2,

$$\left.\begin{aligned} \frac{p_{c1}}{\rho_1} &= \Phi_{c1} \neq 0\,, \\ \frac{p_{c2}}{\rho_2} &= \Phi_{c2} = 0\,. \end{aligned}\right\} \qquad (256)$$

It makes no difference in the results which of these sets of relations we employ, but we must use one or the other.

With this understanding we obtain from equation (252) or (254) the elevation of any point on a fluid interface at which the corresponding potentials are known. For example, if we consider a water-air interface and define our potentials on the air side of the interface, letting air be the fluid of density ρ_1 and water that of density ρ_2, then both Φ_1 and Φ_{c1} are sensibly equal to zero, and

$$\left.\begin{aligned} z &= \frac{1}{g} \cdot \frac{\rho_2}{\rho_2 - \rho_1} \cdot (\Phi_2 - \Phi_{c2}) \\ &= \frac{1}{g} \cdot \frac{\rho_2}{\rho_2 - \rho_1} (gh_2 - gh_{c2}) \cong h_2 - h_{c2}\,, \end{aligned}\right\} \qquad (257)$$

where h_2 is the customary manometer height of the water at the interface in a tube of large enough diameter that capillary forces are negligible and $-h_{c2}$ is the capillary rise due to the forces of the surface tensions between air, water, and the medium at the given point. Also, $\rho_2/(\rho_2 - \rho_2) \cong 1$.

The angle of slope of this interface is obtainable by differentiation:

$$\sin \alpha = \frac{\partial z}{\partial s} = \frac{1}{g} \cdot \frac{\rho_2}{\rho_2 - \rho_1} \left(\frac{\partial \Phi_2}{\partial s} - \frac{\partial \Phi_{c2}}{\partial s} \right) \cong \frac{\partial h_2}{\partial s} - \frac{\partial h_{c2}}{\partial s}\,. \qquad (258)$$

If the medium is homogeneous and the water static, h_2 = constant, h_{c2} = constant, and the surface will be horizontal. If the medium is heterogeneous and the water static, h_2 = constant, $h_{c2} \neq$ constant, and the surface will be undulatory. In coarse gravel and open spaces $h_{c2} \cong 0$, so that the elevation of the surface becomes equal to h_2. When the water is not static but flows through a homogeneous medium, h_{c2} = constant and

$$\frac{\partial h_{c2}}{\partial s} = 0 \qquad \text{and} \qquad \frac{\partial z}{\partial s} \cong \frac{\partial h_2}{\partial s} .$$

Hence, the slope of the surface when the capillarity is constant is related to the rate of flow in the same manner as when the capillarity is zero.

For the more general case the elevation z of the interface is obtainable from equation (254) when one of the fluids is static. Let the ρ_2-fluid be static and the potentials taken on its side of the interface. Then

$$\Phi_2 = \Phi_{20} = \text{constant} ,$$

$$\Phi_{c2} = 0 ,$$

and

$$\left. \begin{aligned} z &= \frac{1}{g}\left[\frac{\rho_2}{\rho_2 - \rho_1} \cdot \Phi_{20} - \frac{\rho_1}{\rho_2 - \rho_1}(\Phi_1 - \Phi_{c1})\right] \\ &= \frac{\rho_2}{\rho_2 - \rho_1} \cdot h_{20} - \frac{\rho_1}{\rho_2 - \rho_1}(h_1 - h_{c1}) , \end{aligned} \right\} \qquad (259)$$

where h_{20} = constant is the manometer height of the ρ_2-liquid, h_1 is that of the ρ_1-liquid, and h_{c1}, which may be either positive or negative, is the additional manometer rise of the ρ_1-fluid due to capillarity. When both fluids are static and the medium homogeneous, the interface is horizontal; when the medium is heterogeneous, it is undulatory.

For the most general state of stationary flow the angle of slope of the interface is obtainable from equation (254) by differentiation:

$$\sin \alpha = \frac{\partial z}{\partial s} = \frac{1}{g}\left[\frac{\rho_2}{\rho_2 - \rho_1} \cdot \frac{\partial \Phi_2}{\partial s} - \frac{\rho_1}{\rho_2 - \rho_1}\left(\frac{\partial \Phi_1}{\partial s} - \frac{\partial \Phi_{c1}}{\partial s}\right)\right] , \qquad (260)$$

where the potentials are taken on the ρ_2-side of the interface. For a homogeneous medium, $\partial\Phi_{c1}/\partial s = 0$ and equation (260) reduces to equation (190) derived for zero capillarity.

From the foregoing we see that the effect upon a fluid interface produced by capillarity is principally one of a greater or less distortion with respect to the configuration of the same interface for zero capillarity. While the interfacial pressure differences due to capillarity between pairs of liquids is, in most cases, less than that for

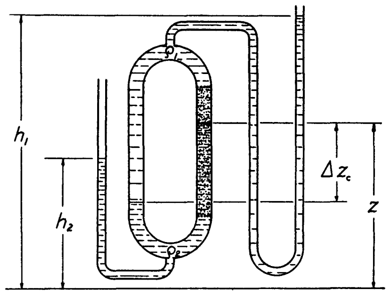

FIG. 37.—Equilibrium levels of the interface between two static liquids of different densities in an open space and in sand.

water-air, the vertical displacements are equal to the manometer rises h_{c1}, or h_{c2}, magnified by the factor $\rho_1/(\rho_2 - \rho_1)$ or $\rho_2/(\rho_2 - \rho_1)$, depending upon which fluid is taken for reference, which commonly is large compared with unity.

Also, in extremely fine-grained rocks whose permeabilities for ordinary purposes may be negligible, the capillary pressures may reach several atmospheres. In the course of geologic time, between otherwise static bodies, this interface will migrate toward a configuration of equilibrium. The vertical distance of such a migration from the corresponding position for $p_c = 0$ would be

$$z_c = +\frac{1}{g} \cdot \frac{\rho_2}{\rho_2 - \rho_1} \cdot \Phi_{c2} = +\frac{\rho_2}{\rho_2 - \rho_1} \cdot h_{c2} = +\frac{1}{g} \cdot \frac{p_{c2}}{\rho_2 - \rho_1}; \quad (261a)$$

or

$$z_c = -\frac{1}{g} \cdot \frac{\rho_1}{\rho_2 - \rho_1} \cdot \Phi_{c1} = -\frac{\rho_1}{\rho_2 - \rho_1} \cdot h_{c1} = -\frac{1}{g} \cdot \frac{p_{c1}}{\rho_2 - \rho_1}, \quad (261b)$$

corresponding to whether the fluid considered is that of density ρ_2 or of density ρ_1, respectively. In either case the capillary pressure is that *against* the fluid considered, so that $p_{c2} = -p_{c1}$. With $\rho_1/(\rho_2 - \rho_1) = 10$ this would amount to approximately 100 meters per atmosphere. Apparently phenomena of this sort have played a significant role in the accumulation of petroleum and are important in its extraction.

For the dynamic case for which one or both fluids are in steady motion through a homogeneous medium, it is not to be assumed that the interface is simply displaced vertically by the amount $(1/g) p_c/(\rho_2 - \rho_1)$ above or below its position for $p_c = 0$. In this case any shift of the interface causes corresponding changes in the flow patterns and cross-sectional areas of the stream tubes. This results in a shift along the flowlines of the equipotential surfaces, and all of these several changes affect the new position of equilibrium, which, when the potential gradients are large, can only be obtained from equation (254) or its equivalent.

THE WATER TABLE

One additional point should be given attention before leaving the subject of capillarity. If, at any given point in a region underlain by permeable rocks, a vertical, uncased well is dug, it eventually reaches a depth at, and below which, water flows into the hole and stands in static equilibrium. It has become standard usage in ground-water hydrology to refer to such a static air-water interface in an open hole as the "water table" at that point. Hence, the water table represents an elevation below which water flows freely into an uncased hole and above which no such flow occurs.

This fact has led, naturally enough, to the conception, also of wide currency among ground-water hydrologists, that the water table underground forms a bounding surface between a lower region which is completely saturated with water and an upper region in which the saturation is incomplete. The water table is in this manner conceived to be the upper surface (except where this is formed by impermeable rocks) of a "zone of saturation" and the lower surface of a "capillary fringe"—a zone (presumably in which the saturation is incomplete) in which the water is sustained by capillarity.

Of late it has been recognized that flow occurs within this "capillary fringe" zone as well as within the "zone of saturation," and investigations have been suggested to determine the manner of this flow.

Thus, while our facts concerning the water table are limited to observations made within holes or other large voids in the ground, it is clear that the conceptions regarding the water table are not so limited; and, in fact, the water table is conceived to be a continuous surface (except where the rocks are impermeable) whose elevation coincides at every point with the position the air-water interface would assume in an uncased open hole *were one placed at that point.*

Our main interest here centers upon the question of what is the physical significance, if any, of such a surface in an underground region away from holes or other voids larger than the pore spaces of the medium. Does it, in fact, form a boundary between a lower region completely saturated with water and an upper region incompletely saturated? Does it represent the air-water interface? Does it form any kind of a recognizable boundary that one could distinguish visually in a permeable region bounded by a vertical glass plate?

The answers to all of these questions are implicitly contained in the discussion of capillarity that we have just made and are experimentally verifiable by means of very simple apparatus containing sand, water, and air, with one or more vertical glass walls for visual observation and with suitable manometer taps.

The answer is that only one unclosed and continuous air-water interface exists, on either side of which there may or may not occur air bubbles trapped in the region otherwise saturated by water, and water meniscules similarly trapped and held in the region otherwise saturated by air. Furthermore, the isolated air bubbles and water meniscules represent unstable configurations which will disappear spontaneously, the air bubbles by solution and the water meniscules by evaporation, unless renewed by processes foreign to those presently considered.

For simplicity, consider the static example afforded by a large laboratory graduate completely filled with sand and partly filled with water. The air-water interface will be clearly visible (especially if the water be colored); and the minutest inspection will not indicate the position of the water table, because no boundary between a zone of complete saturation and one of incomplete saturation (other than the air-water interface itself) will be discernible; neither will a surface of any other physical discontinuity appear.

If, on the other hand, we tap manometer tubes into any part of the region below the air-water interface, they will all register the same elevation h corresponding to a region of constant potential, gh. Furthermore, this elevation h will be precisely the same as the elevation of the water table as obtained from a "well" driven inside the graduate, and will be less than the elevation z_{12} of the air-water interface by an amount h_c, which is the decrease of the manometer height owing to the capillary pressure from the value it would assume were the capillary pressure zero.

If we now consider the pressure field in the region below the air-water interface, we shall find, in this static case, that we have an infinite family of horizontal isobaric surfaces with the pressure decreasing upward at the rate $\rho_2 g$ per unit of length, where ρ_2 is the density of the water. Furthermore, the height of the water in the manometer tubes and in the "well" will coincide precisely with the isobaric surface $p = 1$ atmosphere. (In this and the paragraphs immediately following, the term "1 atmosphere" is to be understood to mean prevailing atmospheric pressure, and not "1 standard atmosphere.") Below this surface the pressure will be greater than 1 atmosphere; and above, between it and the air-water interface, the pressure will be less than 1 atmosphere.

From this it is clear that the water table in this static case is nothing other than the isobaric surface, $p = 1$ atmosphere; and the so-called "zone of saturation" comprises the region throughout which $p > 1$ atmosphere. Likewise, the "capillary fringe" is the region between the isobaric surface, $p = 1$ atmosphere, and the air-water interface, characterized by $p < 1$ atmosphere.

While this simpler static case is employed for purpose of illustration, the above results are nowise altered when a potential gradient is impressed upon the system and the water made to flow. The isobaric surface, $p = 1$ atmosphere, still is the water table and represents in underground regions no physical discontinuity of any kind.

The equipotential surfaces cross this isobaric surface without interruption and extend to the air-water interface across which they refract into the horizontal. The fluid flow obeys precisely the same laws in the region where $p < 1$ atmosphere as in that characterized by $p > 1$ atmosphere.

On the other hand, if the upper surface of the "zone of saturation" be regarded to be coincident with the air-water interface—as the term itself suggests that it should be—then it does not coincide with the water table but stands consistently above that surface by a height h_c; and a "capillary fringe" does not exist.

Hence, in either case we are left with no alternative but to discard the conception of the water table as a surface of discontinuity between a "zone of saturation" and a "capillary fringe" having fundamentally different physical characteristics as a misleading fiction.

THE EFFECT OF TEMPERATURE AND PRESSURE UPON THE FLUID PROPERTIES

Aside from surface-tension phenomena, the fluid properties which enter into our equations are the density and viscosity. For a single homogeneous fluid both these properties are single-valued functions of the temperature and pressure. For gases within the range of validity of the gas laws the viscosity and density are determinable from the kinetic theory of gases. For gases outside this range and for liquids these properties must be obtained experimentally.

In the underground regions we are interested in there exists not only a pressure field but a thermal field as well, giving rise to the two families of field surfaces: isobaric surfaces and isothermal surfaces. Each point underground will be located upon one of each of these surfaces, and at that point the values of the density and viscosity of the fluid in question will be those appropriate to the temperature and pressure. This will give rise to two scalar fields—one for viscosity and one for density—each with its appropriate equiscalar surfaces.

Setting

$$\left.\begin{aligned} \eta &= f(T,p) \,, \\ \rho &= F(T,p) \,, \end{aligned}\right\} \tag{262}$$

and considering that T and p are both functions of position underground, then along any path s

$$\left.\begin{aligned}\frac{d\eta}{ds} &= \frac{\partial\eta}{\partial T}\cdot\frac{dT}{ds}+\frac{\partial\eta}{\partial p}\cdot\frac{dp}{ds}\,,\\ \frac{d\rho}{ds} &= \frac{\partial\rho}{\partial T}\cdot\frac{dT}{ds}+\frac{\partial\rho}{\partial p}\cdot\frac{dp}{ds}\,.\end{aligned}\right\} \tag{263}$$

Integration of equations (263) along the path s from an initial position s_0 to a final one s gives the corresponding values of η and ρ at the final position. Thus,

$$\left.\begin{aligned}\eta &= \eta_0+\int_{s_0}^{s}\left(\frac{\partial\eta}{\partial T}\cdot\frac{dT}{ds}+\frac{\partial\eta}{\partial p}\cdot\frac{dp}{ds}\right)ds\,,\\ \rho &= \rho_0+\int_{s_0}^{s}\left(\frac{\partial\rho}{\partial T}\cdot\frac{dT}{ds}+\frac{\partial\rho}{\partial p}\cdot\frac{dp}{ds}\right)ds\,.\end{aligned}\right\} \tag{264}$$

For gases, both of these equations have to be taken into account; but for liquids, which are our principal concern here,

$$\frac{\partial\rho}{\partial T}\cong 0\,;\qquad \frac{\partial\rho}{\partial p}\cong 0\,,$$

so that $\rho \cong \rho_0$ or is sufficiently nearly constant that, for most purposes, it may be so regarded. The viscosity for liquids, however, cannot be considered constant within the range of temperatures and pressures encountered by deep wells.

In regions of mild topographic relief and free from thermal anomalies, such as hot springs and volcanic activity, the isothermal surfaces are very nearly horizontal. In similar regions, except locally, where high potential gradients may exist, the isobaric surfaces are nearly horizontal. Under these circumstances, to a very good approximation the pressure and temperature are functions of depth only, and isothermal and isobaric surfaces are substantially parallel. Also, the increase of each of these quantities with depth is approximately linear, and we may set

$$p = -\rho g z + \text{constant} \tag{265}$$

$$T = -\Theta z + \text{constant} \tag{266}$$

where z is the elevation of the point (negative of the depth), ρ the density of water, and Θ the geothermal gradient. In most deep wells it has been found that the fluid pressure is approximately that given by equation (265), though certain anomalous cases where the pressure is much greater than this are known.

We may now consider the path of integration of equation (264) to be vertical and substitute z for s. From equations (265) and (266)

$$\frac{dp}{dz} = -\rho g \,, \tag{267}$$

$$\frac{dT}{dz} = -\Theta \,. \tag{268}$$

Introducing these into the equation (264) for η then gives

$$\eta_z = \eta_0 - \int_{z_0}^{z} \frac{\partial \eta}{\partial T} \cdot \Theta \cdot dz - \int_{z_0}^{z} \frac{\partial \eta}{\partial p} \cdot \rho g \cdot dz \,. \tag{269}$$

By measurements in deep wells Θ is found to vary somewhat widely from one location to another, but for a given well it is approximately constant, an average value being about 3° C. per 100 meters, or 3×10^{-4} °C. cm.$^{-1}$. Also, ρg is approximately constant with the numerical value of about 10^3 gm. cm.$^{-2}$ sec.$^{-2}$.

The values of $\partial\eta/\partial T$ and $\partial\eta/\partial p$ for water are obtainable from Bridgman's[13] experiments, which show that the magnitude of the term under the second integral is less than 10^{-3} of that under the first integral. The second integral may therefore be entirely disregarded for water, leaving

$$\eta_z = \eta_0 - \int_{z_0}^{z} \frac{\partial \eta}{\partial T} \cdot \Theta \cdot dz = \eta_0 + \int_{T_0}^{T} \frac{\partial \eta}{\partial T} \cdot dT \,, \tag{270}$$

where T_0 is the temperature under atmospheric pressure. Computing the approximate temperature as a function of depth, equation (270) gives the viscosity of water as a function of depth. Inserting this into the equation for the specific conductivity then gives the manner in

[13] P. W. Bridgman, "The Effect of Pressure on the Viscosity of Forty-three Pure Liquids," *Proc. Amer. Acad. Arts and Sci.*, Vol. LXI (1926), pp. 57–99.

which that increases with depth for a medium of uniform permeability. These values are computed to depths of 3,000 meters and are given in Table 2, which shows that, when dealing with the flow of water, the decrease of viscosity with depth is equivalent to a corresponding increase of permeability at constant temperature and must be allowed for.

TABLE 2

VARIATION OF VISCOSITY AND SPECIFIC CONDUCTIVITY FOR WATER WITH DEPTH

Depth in Meters	Temperature at °C.	Viscosity of Water Poises	Relative Specific Conductivity
0.........	10	1.308×10^{-2}	1
100.........	13	1.207×10^{-2}	1.08
200.........	16	1.112×10^{-2}	1.18
300.........	19	1.032×10^{-2}	1.27
400.........	22	0.961×10^{-2}	1.36
500.........	25	0.894×10^{-2}	1.46
1,000.........	40	0.656×10^{-2}	2.00
2,000.........	70	0.406×10^{-2}	3.22
3,000.........	100	0.284×10^{-2}	4.61

ROTATIONAL FIELD OF FORCE

In the most general case, where both η and ρ vary appreciably with temperature and pressure and where the isothermal surfaces are not parallel to the isobaric surfaces, the surfaces of equal density—isopycnic surfaces—will intersect the isobaric surfaces, and

$$\oint \frac{dp}{\rho} \neq 0 .$$

When this occurs, no potential is possible, strictly speaking. What happens physically in such a case is that the field of force has a rotational component which tends to set up convectional flow in the fluid. This kind of situation arises from unequal heating in different parts of the flow field, causing the fluid at the same pressure and different temperatures to have different densities. Such a state of flow is not a violation of the conservation of energy but an evidence of an interchange between thermal and mechanical energy, producing a heat engine. The rotational force is commonly superposed upon a potential field of force. In ground-water, while the rotational force field exists in certain cases, owing to the approximate constancy of the density of water, it is so small, compared with the potential force field, as to be negligible.

COMPARISON OF PRESENT THEORY WITH METHODS USED BY OTHERS

This concludes our formal development of theory. We have attempted to establish upon a physically sound basis a theory of the motion of a homogeneous fluid through permeable media, capable of extension to include the flow of fluids of the most general kind but here restricted principally to liquids and the problems of ground water. In so doing, we have intentionally confined ourselves to two important restrictions: that the flow dealt with shall be either continuously or statistically steady during the time interval considered and that the medium be isotropic.

The reasons for the first of these restrictions is to avoid serious mathematical difficulties encountered in attempting to deal with transient phenomena. These involve the treatment of fields whose properties are functions both of position and of time. While these problems can be set up simply by adding time to the variables herein considered, their solution is far more difficult than the corresponding ones with steady motion. Besides, most ground-water problems can be reduced to problems of steady states of flow.

Recently Theis[14] has studied the nonsteady flow around a well with very promising results, using an analogy to heat diffusion. He has assumed, erroneously, however, that pressure is the hydraulic analogue of temperature.

The restriction that the medium be considered isotropic is partly to avoid the mathematical difficulties inherent in the treatment of anisotropic phenomena and partly in consideration of the fact that nearly all permeability data are taken on the basis of the same assumption. The effect of anisotropy is to produce a distortion in the stream-flow pattern with respect to the corresponding pattern for isotropic conditions for the same potential field. For isotropic media

[14] Charles V. Theis, "The Relation between the Lowering of the Piezometric Surface and the Rate and Duration of Discharge of a Well Using Ground Water Storage," *Trans. Amer. Geoph. Union,* 1935, pp. 519–24.

the flowlines are parallel to grad Φ; for anisotropic media they make an angle with this direction, the magnitude of which depends upon the degree of anisotropy and upon the direction of grad Φ with respect to the permeability axes of the media.

It is important now that we compare the present treatment with those of other authors. It is to be expected that the points of agreement will far exceed those of disagreement, but it is the latter which are of greatest importance, because they are the ones which lead to disagreement in the conclusions drawn and must therefore embody some of the errors that have been committed.

VELOCITY POTENTIAL

Since the classical studies of Slichter most of the authors who have studied the motion of ground water analytically have sensed that some sort of a potential function was involved and have attempted to formulate this function. The approach to this problem has usually

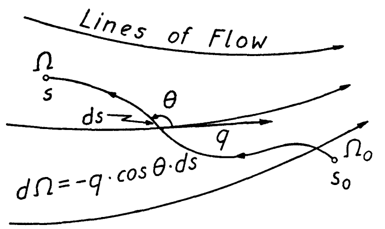

FIG. 38.—The line-integral by means of which a velocity potential Ω is defined

been made by starting with the conceptions of classical hydrodynamics and assuming that the potential function sought was a *velocity potential.*

In this connection it should be explained that a velocity potential is the line-integral of velocity along a path in the flow field, just as a force potential is the line-integral of a force. Hence, if a velocity potential exists in a field of flow, its value at any point is

$$\Omega = \Omega_0 - \int_{s_0}^{s} q \cos\theta \cdot ds , \qquad (271)$$

where Ω_0 is the velocity potential of the initial and Ω that of the final point, q the magnitude of the specific discharge, or velocity vector, and θ the angle between direction of flow and the displacement ds.

Slichter stated that the hydrostatic pressure had the property of a velocity potential in ground-water motion and that isobaric surfaces were accordingly normal to the flowlines. Later, however, he violated his own proposition by dealing with a problem in which he showed the isobaric surfaces to be at all angles from normal to tangential to the related flowlines.

Since that time it appears to have been more often the rule than the exception for authors to employ pressure as a potential function and to write what has been represented to be Darcy's law in the form

$$q_s = -K' \cdot \frac{\partial p}{\partial s},$$

which has already been shown to be invalid.

Even when Darcy's law has been correctly stated in the form

$$q_s = -K \cdot \frac{\partial h}{\partial s},$$

one of the most common errors has been to assume that different manometer tubes for which the heights h are the same must be terminated upon the same isobaric surface.

Dachler, who employs the correct form of Darcy's law, states that

$$q_x = -K \cdot \frac{\partial h}{\partial x} = -\frac{\partial(Kh)}{\partial x}, \quad \text{etc.,}$$

and concludes that (Kh) has therefore the properties of a velocity potential.

Muskat first sets up Darcy's law in the erroneous form

$$q_s = -\frac{k}{\eta} \cdot \frac{\partial p}{\partial s},$$

but later, for the study of general field problems, employs

$$\Omega = \frac{k}{\eta}(p + \rho g z)$$

as a velocity potential, from which

$$q_x = -\frac{\partial \Omega}{\partial x}, \quad \text{etc.}$$

The same assumption has been stated explicitly by Gardner, Collier, and Farr,[15] who say in their opening sentence: "As was pointed out originally by Slichter, the generalization of Darcy's law is nothing more nor less than the criterion for the existence of a velocity potential for the macroscopic velocity of moisture in the soil." They then state that the quantity

$$\sigma\Phi = \sigma\left(gz + \frac{p}{\rho}\right)$$

is a velocity potential.

The trouble inherent in all approaches to the motion of ground water by means of the employment of a velocity potential arises from the basic assumption that a velocity potential exists. The necessary and sufficient condition for the existence of a velocity potential is that for all closed paths in a singly connected region

$$\oint q \cos\theta \cdot ds = 0 . \qquad (272)$$

If this integral is not zero, no velocity potential exists, and any employment of a function supposed to be a velocity potential for such a region is necessarily erroneous.

In this connection it is important to note that in classical hydrodynamics, where the conception of a velocity potential originated, the only fluids to which it is applicable are idealized frictionless fluids. In real fluids for which the friction is not negligible the line integral around a closed curve ceases to be zero, and the motion is not derivable from a velocity potential.

From this fact alone the assumption that ground water possesses a velocity potential might appear to be of doubtful validity, for here friction is a dominant influence. On a microscopic scale there is no question that a velocity potential cannot exist, but whether this is true macroscopically requires further investigation.

One test of the validity of this assumption is provided by the rectilinear flow of water between vessels, through a prism of sand. Let the prism be divided into two sections by a plane parallel to its axis,

[15] Willard Gardner, T. R. Collier, and Doris Farr, "Fundamental Principles Governing the Control of Ground-Water," *Trans. Amer. Geoph. Union*, 1934, pp. 563–66.

one section containing sand of permeability k_2 and the other of k_1, where $k_2 > k_1$. The flowlines will be parallel to the sand interface; but in the sand of permeability k_2, the specific discharge will have the magnitude q_2, and in that of permeability k_1 the magnitude of the discharge will be q_1.

Let A be a point at the end of the sand prism in the downstream vessel, and B a corresponding point in the upstream vessel. Let us assume that a velocity potential exists, and let Ω_A be its value at A. Then at B

$$\Omega_B = \Omega_A - \int_A^B q \cos \theta \cdot ds \,. \tag{273}$$

Now let us evaluate this integral along two separate paths: (1) From A the path follows the normal to the flowlines until a flowline through

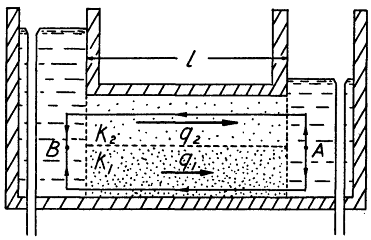

FIG. 39.—The velocity line-integral from A to B by two different paths traversing sands of different permeabilities.

the k_1 region is encountered. It then follows this flowline until the normal passing through B is reached; thence along that normal to B. (2) For the second path we proceed, as before, along a flowline-normal from A, then along a flowline in the k_2 region until the normal through B is reached, and finally along this to the point B. In both cases the distance l along the flowlines is the same. When the flowlines are followed, θ is 180°, the cosine of which is -1; and when the flowline-normals are followed, θ is 90°, whose cosine is zero.

By these separate paths the values of the potential at B are

$$\left.\begin{aligned} \Omega_{B1} &= \Omega_A + q_1 \cdot l \,, \\ \Omega_{B2} &= \Omega_A + q_2 \cdot l \,, \end{aligned}\right\} \tag{274}$$

from which

$$\Omega_{B_2} - \Omega_{B_1} = (q_2 - q_1)l \neq 0 \qquad (275)$$

and

$$\oint q \cos\theta \cdot ds \neq 0 \,.$$

This shows us clearly that in this special case no velocity potential is possible and that the value of the assumed velocity potential at B depends upon the path traversed.

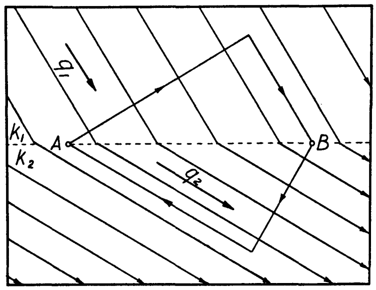

FIG. 40.—Line-integral around a closed path traversing regions of different permeabilities.

A more general case is encountered if we take the points A and B upon the interface between two sands of permeabilities k_1 and k_2 across which the flowlines pass obliquely. Starting from A we traverse the flowline-normal in the k_1 region until the flowline passing through B is reached. We follow this to B and then follow the flowline-normal through B in the k_2 region until the flowline through A is encountered along which the return to A is made.

If we let γ_1 and γ_2 be the angles between the flowlines and the surface-normals in the two regions, then from the tangent law of refraction, when $k_2 > k_1$,

$$\gamma_2 > \gamma_1 \,, \qquad \text{and} \qquad q_2 > q_1 \,.$$

The line-integral around the path described is therefore

$$\oint q \cos\theta \cdot ds = \pm\overline{AB}(q_2 \sin\gamma_2 - q_1 \sin\gamma_1) \neq 0 \,, \qquad (276)$$

so that for this case, too, no velocity potential exists.

While it is possible by the extension of the above method of analysis to arrive at a generalization, we can do this better by returning to our fundamental equations. The expressions for a velocity potential employed by Dachler, by Muskat, and by Gardner, Collier, and Farr are all equivalent and are related to the force potential Φ, as herein employed, in the following manner:

$$\Omega = Kh = \frac{k}{\eta}(p + \rho g z) = \sigma\Phi . \tag{277}$$

From the definition of a velocity potential, if one exists,

$$\mathbf{q} = -\text{grad}\,\Omega = -\text{grad}\,(\sigma\Phi) ,$$

which is none other than equation (136) that has already been shown to be invalid as a general expression of Darcy's law.

The same result is obtained if we perform the line-integral around a closed path:

$$\oint q \cos\theta \cdot ds = -\oint \sigma \cdot \frac{\partial\Phi}{\partial s} \cdot ds , \tag{278}$$

which, if a velocity potential exists, must be zero for all possible paths.

But we know already that for all cases for which Φ is determinate, namely, when the density of the fluid is a function of its pressure only so that its isopycnic and isobaric surfaces are mutually parallel,

$$\oint \frac{\partial\Phi}{\partial s} \cdot ds = 0 . \tag{279}$$

This property of Φ is entirely independent of any properties of the medium, whether homogeneous or inhomogeneous, isotropic or anisotropic, or whether the closed path is confined to the region of the fluid concerned or makes excursions into regions occupied by other fluids. Hence, in order for equation (278) to be zero, it is necessary that σ be eliminated from the integrand, which is only allowable provided the value of σ is independent both of position in the field and of the direction of the flow. Since $\sigma = k\rho/\eta$, these conditions are only satisfied provided $k\rho/\eta$ = constant. It is also necessary that the value of k be independent of the direction of the flow, that is, that the medium be isotropic. Since k, ρ, and η are all independently variable, this requires that all three of them separately must remain constant.

Hence, a velocity potential exists only for fields of flow involving a fluid of constant density and viscosity and a medium which is homogeneous and isotropic throughout. Since these several conditions obtain only in specially set up laboratory problems involving liquids, and are not even approximately realized when fluids of variable density and viscosity are employed, or in ground-water problems involving wide ranges of the permeability, it is clear that the velocity-potential conception of Darcy's law is an inadequate one. Physically, besides possessing an extremely narrow range of validity, the velocity potential conception has only kinematical significance and gives one no insight whatever into the dynamical properties of the flow; mathematically, it has no advantages over the more general dynamical expression here derived since it represents only a special case of the latter. The dynamical expression, on the contrary, is of unlimited validity, being applicable to the flow of fluids of variable density and viscosity through media of space variable permeability. For anisotropic media this validity is still retained, provided the permeability parameter be expressed in the form of a tensor in the place of the scalar k of isotropic media.

THE PIEZOMETRIC SURFACE

Another device that is extensively employed in ground-water practice is the *piezometric surface.* Frequently there exists at depth a stratum of rock such as sandstone or porous limestone of high permeability, overlain by rocks of slight permeability. Wells terminated in these upper rocks yield sparingly, while those reaching the stratum of high permeability, or aquifer, yield more abundantly. This fact focuses attention upon the hydrologic properties of the aquifer, and wells reaching it can be employed as manometer tubes in addition to their function as wells.

In a particular vertical well terminated upon the top of such an aquifer the water will rise to a static elevation h above the standard datum. Now, if we imagine a large number of such wells to be drilled to the top of the aquifer and used as vertical manometer tubes, the heights h will vary with position, but at any given time they will all lie upon a smooth surface, $h(x,y)$, where x and y are the horizontal co-ordinates of the wells. This surface is what is known as the "piezometric surface."

Now it is clear that at each well the height h gives the potential $\Phi = gh$ at the lower terminus of that well, and a complete knowledge of the piezometric surface of any given stratigraphic horizon over a given area enables one to determine completely the potentials at all points on that horizon over the corresponding area. Lines of equal potential can then be drawn upon the underground horizon; and from these, by means of Darcy's law, if the permeability of the aquifer is known, the component of flow tangential to its upper surface can be determined.

While this is a procedure of great practical importance, it also has serious limitations which must be thoroughly understood if important errors are to be avoided. For one thing, it must be kept clearly in mind that the ground-water potential field and flow field is a three-dimensional field occupying all the space from the air-water interface down to indefinite depths, and all that we have done is to determine the value of this three-dimensional field upon an arbitrary two-dimensional surface through that field. The equipotential lines on this surface are, therefore, but the traces of the equipotential surfaces that have been intersected by it.

In actual practice it is customary to use the piezometric surface, not as a means of obtaining the potential of points on the underground surface, but directly, assuming that the component of the potential gradient on the underground surface is proportional to the slope of the piezometric surface itself, or that

$$\frac{\partial h}{\partial s} = \frac{\partial h}{\partial r}, \qquad (280)$$

where s is a length on the underground surface and r its vertical projection upon a horizontal surface. This is strictly true if the bedding surface considered is horizontal, but if s has an angle of slope α, then

$$s = \frac{r}{\cos \alpha}$$

and

$$\frac{\partial h}{\partial s} = \frac{\partial h}{\partial r} \cdot \cos \alpha . \tag{281}$$

Hence, such an assumption is not valid for angles of dip greater than about 3°, though in most cases the dips actually dealt with are smaller than this.

A more serious danger lies in the tacit assumptions that are commonly made when computing flow from the geometry of the piezo-

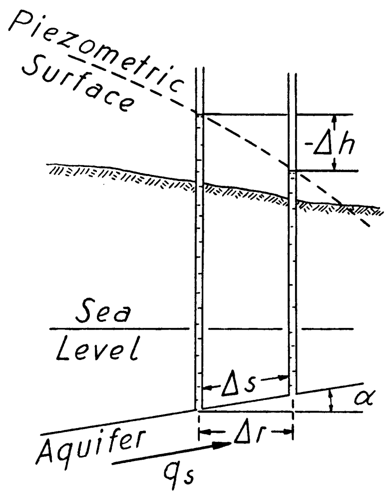

FIG. 41.—Section showing tilted aquifer and piezometric surface

metric surface. These are that the aquifer is bounded above and below by rocks so impermeable that the normal components of the flow across these bounding surfaces are negligible, so that the tangential components are essentially the total flow, and the flowlines from top to bottom of the aquifer are parallel to its bounding faces. The postulated flow is, therefore, parallel to a single plane, and the potentials on one face of the aquifer are the same as those on all other planes parallel to this.

Again, in many instances these assumptions are good approximations; but they must be made knowingly and with caution, because there are many other instances where they are not even approximately correct. If, for example, into a horizontal laminar aquifer whose upper and lower surfaces are completely impermeable, two separate wells with equal discharges are terminated—one as a point-sink on the upper boundary of the aquifer, and the other as a line-sink extending from top to bottom—the potentials along the upper surface of the aquifer and the corresponding piezometric surfaces will be quite different in the two cases. In the case of the line-sink we shall deal with a problem of plane radial flow with logarithmic potential. For the point sink, in its immediate vicinity the flow will be hemispherically radial and the potential will increase with the inverse first power of the distance. At greater distances this will transform gradually into plane flow with logarithmic potentials. While the potentials are obtainable from the piezometric surface in both cases, the methods of analysis to be used in computing the flow are quite different.

It is not uncommon to encounter on a piezometric surface so-called piezometric "highs"—the large southern high, the Tampa high, and others on the piezometric map based on the top of the Ocala limestone in Florida,[16] for example. These consist of large "mounds" on the piezometric surface or of regions underground where the equipotential lines on the surface of the aquifer enclose areas of higher potential. Since, in such an instance, the component of the flow parallel to the upper surface of the aquifer is radially outward from the high, either of two things must occur: water must be created in the region of the high, or else water must be entering such a region across its upper or lower surfaces, the latter, of course, being the actual situation. Such instances merely show conspicuously that the assumption of impermeable upper or lower boundaries with a zero normal component of flow of an important aquifer is not tenable.

These instances illustrate the caution that must be observed in drawing conclusions from a piezometric surface. It is shown by treatises dealing with the mathematical theory of potential functions that in a region whose geometry and fixed physical properties—in this case permeability—are known completely, it is possible to determine the potential at all points in the field when its value at all

[16] V. T. Stringfield, "The Piezometric Surface of Artesian Water in the Florida Peninsula," *Trans. Amer. Geoph. Union*, 1935, pp. 524–29.

points upon a surface passing through that field is known. This is precisely the problem involved when a piezometric surface is given. In very simple cases, such as the regional flow through an aquifer, simple calculations suffice. When dealing with local anomalies, however, such as the disturbances due to wells, springs, canals, and the like, the method of calculation must be correctly adapted to the geometry of the particular problem considered, otherwise the results obtained will not be reliable.

Among those who employ the piezometric surface as a device for determining regional flow, many of these difficulties are not unrecognized; and on various occasions the statement has been made that for problems involving flow in three dimensions an infinity of piezometric surfaces would be required.

While such a procedure may not be impossible, the difficulties inherent in it are rather formidable. Assuming the permeability throughout the field to be known completely, all that is directly determinable (without recourse to extraneous considerations) from the piezometric surface corresponding to a given plane intersecting the field of flow are the components parallel to that plane of the flow vector at all points in the plane. To obtain the resultant flow vector at any point, three non-coplanar vector components must be known. These necessitate the use of three piezometric surfaces corresponding to three nonparallel planes intersecting at the given point. Then, for all points in the field of flow it would be necessary to have the ensemble of piezometric surfaces corresponding to three infinite families of planes, one family parallel, respectively, to each of the nonparallel planes through the initial point—a triple infinity in all.

But, before we are able to construct these families of piezometric surfaces, we must first know the value of h, and hence of $\Phi = gh$, at every point in the field. If we know this, however, we can solve the problem directly from the known potential gradients by means of Darcy's law, and no piezometric surfaces are required.

THE CONDITION FOR ARTESIAN WATER

At this point it is pertinent to inject a comment regarding the conditions for artesian wells. Since water always flows from regions of higher to those of lower potential, if we terminate one end of an open tube at a point P_1 in a ground-water field of flow, and the other end at P_2 either inside or outside the field, the water will flow through the tube from P_1 to P_2 provided the potential Φ_1 at P_1 is greater than Φ_2 at the point P_2. The only exception to this occurs when the path traversed by the tube is such that negative fluid pressures are involved, as in the case of too high a syphon.

With this in mind, if we take the point P_2 above the surface of the ground and we wish artesian water at that point, all that is necessary is to find a point P_1 within the ground-water field at which the potential Φ_1 is greater than Φ_2 and connect these two points by an open-ended tube so arranged that no negative pressures exist. Artesian flow will then occur at P_2. If we restrict the location of P_1 to a vertical line through P_2, then it is both necessary and sufficient that a potential greater than Φ_2 exist somewhere along this line beneath the air-water interface if artesian flow is to occur at P_2.

The variety of hydrologic situations that will give rise to potentials below the ground surface larger than those a few feet above is very great. One of these is the familiar textbook illustration of the aquifer that outcrops in the highlands and then passes under the plains beneath an overburden of impermeable strata, a typical example of which is afforded by the Dakota sandstone under the great plains. While this is a sufficient condition for artesian water, it is by no means a necessary one, and its extensive use has obscured a more complete understanding of artesian phenomena.

As a matter of fact, in order to have artesian potentials, an aquifer need not be overlain by impermeable material, and it need not outcrop. Moreover, even if it does outcrop, the outcrop area may just as well be a region of discharge as of intake. All these points, as Paige[17] has pointed out, are illustrated by the Ocala limestone of Florida. Over much of the southern and eastern part of the state the water from this aquifer is under artesian potentials; yet its outcrop area is near sea-level and is a region of discharge. The water enters the formation by downward flow through 500 feet and more of

[17] Sidney Paige, "Effect of Sea-Level Canal in the Gound-Water Level of Florida—A Reply," *Econ. Geol.*, Vol. XXXIII (1938), pp. 647–65.

younger and less permeable formations from regions that are topographically higher than the artesian areas.

If it ever becomes practicable to drill inclined or horizontal wells, artesian water can be obtained at many places where it cannot be obtained from vertical drill holes.

THE PROBLEM OF PERMEABILITY

When Darcy established experimentally by means of the flow of water through filter sands the relationship of equation (8),

$$q = -K \cdot \frac{dh}{dl} ,$$

he remarked that K was a "coefficient depending upon the permeability of the bed" (of sand). By investigating the flow of all manner of fluids through homogeneous and isotropic media, we were able to show in equation (74) that the complete expression for the flow of any kind of fluid under any condition (subject only to the restriction that the velocity be small enough that forces due to inertia may be neglected) is given by

$$q = -Nd^2\rho \cdot \frac{1}{\eta} \cdot g \cdot \frac{\partial h}{\partial l} ,$$

which, when compared with equation (8), shows that

$$K \equiv Nd^2\rho \cdot \frac{1}{\eta} \cdot g = k\rho \cdot \frac{1}{\eta} \cdot g ,$$

where $k \equiv Nd^2$ and is the only one of the above four quantities that is a property of the medium, the remaining factors being properties of the fluid and of the gravity field at the point of experimentation.

Equation (74), while unknown to Darcy himself, is simply a generalization of the special case which he investigated. In conformity with established usage, it is the general equation, rather than the special case, that constitutes "Darcy's law." Consequently, in what follows, when the term "Darcy's law" is used, it shall be understood to signify the general relationship of equation (74) or its equivalent. Any other expression purporting to be Darcy's law but not equivalent to equation (74) can then be shown to be either physically erroneous or else incomplete, that is, an expression which is physically correct as far as it goes, but which is limited in its validity to a narrower range of phenomena.

The identification of equation (8) with equation (74) is only permissible provided we interpret equation (8) to be a complete expression of Darcy's law. Only under this condition are we permitted to expand the parameter K into the component factors k, ρ, $1/\eta$, and g.

In this expanded form of K we are confronted anew with the problem of what combination of these four factors is to represent the essential element of a coefficient of permeability. Darcy's statement that K is a "coefficient depending upon the permeability" of the sand clearly suggests that K is not itself the coefficient of permeability but is a function of that coefficient. What, then, is the coefficient of permeability of the sand?

Attempts to answer this question have led to more disagreement and confusion than perhaps any other problem in ground-water hydrology. As a result, we have at the present time some four or five physically different—that is, dimensionally unlike—quantities all called by the same name: the "coefficient of permeability." In addition to this, a dozen or more different sets of physical units have been prescribed as the proper ones in terms of which the different coefficients are to be measured.

If this kind of confusion is to be eliminated, it can only be done by reaching an agreement as to what concept we wish the term "permeability" to signify, and then determining a coefficient that varies as a function of the selected permeability and with nothing else. The basis of such a coefficient of permeability must necessarily be some combination of the four factors, k, ρ, $1/\eta$, and g, taken from one to four at a time, because these are the only factors that affect the rate of flow. Conversely, the choice of any combination of these four factors uniquely determines a particular definition of permeability.

We could start out by trying to define the different conceptions that might be called "permeability" and then determine what combination of the above factors varies as a function of each, but it is far simpler to proceed in the inverse manner and let the combinations of k, ρ, $1/\eta$, and g, taken from one to four at a time, serve to define the different possible conceptions of permeability. Since there are only fifteen such combinations, all dimensionally unlike, then the maximum number of possible different "permeabilities" is fifteen.

Since the time of Darcy there seems to have been general agreement that permeability is, in part at least, a property of the medium. Thus, if two sands, one coarse and the other fine, are used, and ρ, η, and g are kept constant, for the same value of dh/dl the flow will be at a greater rate through the coarser sand, which will therefore be said to be the more permeable of the two. Since, in this case, only the factor k has been varied, it follows that any coefficient of permeability must contain the factor k as an essential element, whatever else it may embrace besides. This then reduces our possible different conceptions of permeability to the number of combinations that can be formed with k of the factors ρ, $1/\eta$, and g taken from zero to three at a time, or eight in all.

The preponderance of contemporary opinion appears to favor the conception that permeability is a function of the medium only; that is, that for a given medium the permeability is a fixed property independent of the intensity of gravity, of the properties of the fluid flowing through it, or of whether any fluid flows at all. Of the eight combinations with k of the factors ρ, $1/\eta$, and g, the only one satisfying this condition is k itself, which depends only upon the geometrical properties of the medium—its internal shape and its size scale—and has the dimensions $[L^2]$.

Thus, if we choose k as our coefficient of permeability, we uniquely determine that permeability is to be a property of the medium and of nothing else. The converse, however, is not true; if we decide that permeability is to be a property of the medium only, a coefficient signifying this fact is not uniquely determined, because, in addition to k itself, any single-valued function, $\Pi = f(k)$, where Π is the coefficient of permeability, can also be made to serve. All that is necessary is that for any given value of k, corresponding to a particular medium, the value of Π must be uniquely determined. This fact enables us to resolve the anomaly that arises when two widely different coefficients of permeability, both capable of being shown to be properties of the medium only, are employed.

Since there are an unlimited number of single-valued functions of k, we are left with an unlimited number of choices for our final coefficient of permeability. For this choice we are concerned only with the dictates of convenience. Out of all possible single-valued functions of k, by all odds the simplest is the identity

$$\Pi \equiv k\,, \tag{282}$$

whose dimensions are $[L^2]$, and this is the coefficient that we shall here recommend.

There remains only the problem of the choice of suitable units of measurement; and again, if simplicity is to be achieved and confusion avoided, the best procedure is to adhere strictly to the fundamental units and the consistent derived units of either the metric or the English system. In the metric system the unit of k is the square centimeter; in the English system it is the square foot. The conversion factor from the English unit to the metric unit is simply the number of square centimeters per square foot, or 929.0.

Stated in this abstract manner, the square centimeter as a unit of permeability is rather meaningless. This obscurity disappears, however, if we substitute $\partial\Phi/\partial l$ for $g(\partial h/\partial l)$ in equation (74) and solve for k:

$$k = -\frac{q\eta}{\rho\cdot\dfrac{\partial\Phi}{\partial l}}\,.$$

Then, if we equate all members in the term to the right to their unit values, we obtain for the unit value of k:

$$1\,[\mathrm{cm}^2] = 1\left[\frac{[(\mathrm{cm}^3/\mathrm{sec})/\mathrm{cm}^2][\mathrm{gm}/\mathrm{cm\text{-}sec}]}{(\mathrm{gm}/\mathrm{cm}^3)(\mathrm{cm}/\mathrm{sec}^2)}\right]. \tag{283}$$

Expressed in words, a medium has a permeability of 1 $\mathrm{cm.}^2$ when a fluid whose viscosity is 1 poise and whose density is 1 $\mathrm{gm/cm}^3$, flows through it with a specific discharge of 1 $(\mathrm{cm}^3/\mathrm{sec})/\mathrm{cm}^2$ under an impelling force of 1 dyne per gram.

Since we do not have available direct measurements of the permeabilities of sediments expressed in square centimeters from which to obtain an idea of the order of magnitude of this unit, let us postpone further consideration of it until after we have brought up another thread of our discussion.

While it is impracticable to give attention to all the various coefficients of permeability that have been proposed, there are two that merit individual attention; these are that of the Ground Water Division of the United States Geological Survey, which is the one most widely used by ground-water hydrologists, and that proposed originally by Nutting, which is the one most extensively employed by petroleum engineers.

The Ground Water Division unit was suggested by Meinzer[18] and is defined as follows:

> The results of the tests are expressed as a coefficient of permeability, which is based on Darcy's law that the rate of flow varies in direct proportion as the hydraulic gradient. The coefficient of permeability of a material is the rate of flow [of water] in gallons a day, through a square foot of its cross section under a hydraulic gradient of 100 per cent, at a temperature of 60° F. . . . .
>
> The general formula for permeability may be written as follows:
>
> $$P = \frac{qlt}{Tah},$$
>
> in which P is the coefficient of permeability, q the quantity of water, l the length of column of sample, t the correction for temperature, T the time, a the cross-section area of the sample, and h the head.

In later publications of the Ground Water Division it has become customary to employ the equation

$$Q = PIA\,, \tag{284}$$

where Q is the total discharge, P the coefficient of permeability as defined above, I the hydraulic gradient, and A the area of cross section. This is stated to be Darcy's law.

Now if we focus our attention upon equation (284), by converting all of its members except P into the symbols employed here, we bring it into the form

$$q = -P \cdot \frac{dh}{dl}\,. \tag{285}$$

[18] Norah Dowell Stearns, "Laboratory Tests on Physical Properties of Water-bearing Materials," *U.S. Geol. Surv. Water-Supply Paper 596-F* (1928), pp. 121–77.

Since this is stated to be Darcy's law, comparison with equation (74) would lead us to suppose that

$$P \equiv k\rho \cdot \frac{1}{\eta} \cdot g \,, \tag{286}$$

and varies as a function of the properties of the medium, of those of the fluid, and with the intensity of gravity

In this supposition, however, we should be mistaken, for if we restudy the definition of P quoted above, we discover that it applies only to the flow of water at a temperature of 60° F. (and presumably a constant value of g), and to no other fluid. If we wish to know the rate of flow of another fluid—say oil—whose density and viscosity are different from those of water at 60° F., through a medium for which the value of P is known already, not only can we not use equation (284), but we are left without any equation at all.

The equation $Q = PIA$ is therefore an incomplete expression of Darcy's law. To obtain a corresponding equation involving P that is complete, we first translate P into an equivalent expression in terms of k, ρ, η, and g. This we do by noting that for water at 60° F. and a constant value of gravity, ρ, η, and g assume the constant values ρ_0, η_0, and g_0, respectively. In this case equation (284) assumes the form

$$q = -k \cdot \frac{\rho_0 g_0}{\eta_0} \cdot \frac{dh}{dl} \,, \tag{287}$$

which, when compared with the equivalent equation (285), gives

$$P \equiv \frac{k\rho_0 g_0}{\eta_0} \,. \tag{288}$$

Let us now introduce $k(\rho_0 g_0/\eta_0)$ into equation (74) by multiplying the right-hand term of that equation by the quantity

$$\frac{\rho_0}{\rho_0} \cdot \frac{\eta_0}{\eta_0} \cdot \frac{g_0}{g_0}$$

(which is equal to unity). The result obtained is

$$\left.\begin{aligned} q &= -\frac{\rho_0}{\rho_0} \cdot \frac{\eta_0}{\eta_0} \cdot \frac{g_0}{g_0} \cdot \frac{k\rho g}{\eta} \cdot \frac{\partial h}{\partial l} = -\frac{\rho}{\rho_0} \cdot \frac{\eta_0}{\eta} \cdot \frac{g}{g_0} \cdot \frac{k\rho_0 g_0}{\eta_0} \cdot \frac{\partial h}{\partial l} \\ &= -\frac{\rho}{\rho_0} \cdot \frac{\eta_0}{\eta} \cdot \frac{g}{g_0} \cdot P \cdot \frac{\partial h}{\partial l} \,, \end{aligned}\right\} \tag{289}$$

which is valid for any intensity of gravity and for the flow of any fluid whatever. Consequently, this represents a complete expression of Darcy's law and should be the expression employed in all cases where Darcy's law is intended. In a case where we are dealing with the flow of water at 60° F. and with the standard intensity of gravity, we have $\rho = \rho_0$, $\eta = \eta_0$, and $g = g_0$, so that equation (289) reduces, as a special case, to the familiar expression: $Q = PIA$.

A complete expression of Darcy's law involving P does not appear in the literature of the Ground Water Division, and the failure to distinguish between a complete and an incomplete expression of the law has been responsible for most of the confusion concerning the significance of the parameter P. The equation $P = QIA$ is customarily stated without qualification to be Darcy's law.

Returning now to equation (288), we see that k is the sole independent variable. Hence, for every value of k, that of P is uniquely determined. Therefore P is a function of the medium only, and so is a valid coefficient of permeability.

The fact that the dimensions of P are $[LT^{-1}]$ while those of k are $[L^2]$ produces an apparent contradiction. It seems strange that a coefficient dependent only upon the unit of length should involve the units of both length and time. This discrepancy is accounted for by the factor of proportionality, which, while constant, is not dimensionless. When these dimensions are taken into account, we obtain

$$[P] = \left[\frac{\rho_0 g_0}{\eta_0}\right][k] = [LT^{-1}], \tag{290}$$

and the paradox is resolved.

The other coefficient that we wish to consider is that proposed originally by Nutting[19] in 1930 and later adopted by Wyckoff and associates[20] in 1934. In each case an equation of the form

$$q = -\frac{k}{\eta} \cdot \frac{\partial p}{\partial l}$$

was employed, and k was defined to be the coefficient of permeability. Despite the fact that this equation is physically erroneous as an

[19] P. G. Nutting, "Physical Analysis of Oil Sands," *Bull. Amer. Assoc. Petrol. Geol.*, Vol. XIV (1930), pp. 1337–49.

[20] R. D. Wyckoff, H. G. Botset, and D. W. Reed, "Measurement of Permeability of Porous Media," *Bull. Amer. Assoc. Petrol. Geol.*, Vol. XVIII (1934), pp. 161–90.

expression of Darcy's law, owing to the use of pressure as a potential function, the k so defined is identical with that used here.

We return now to the question of the numerical values of k, for the common sediments. These may be obtained directly by solving equation (74) for k when all other quantities of that equation have been determined by measurement. Lacking such data, we may, by means of equation (288), find the value of k indirectly for any sediment for which the value of P is known. Solving equation (288) for k gives

$$k = \frac{\eta_0}{\rho_0 g_0} \cdot P .$$

The measurements of P that are available are expressed in gallons per day per square foot—the units prescribed by Meinzer. To obtain the value of k in square centimeters, we must first convert P into the metric units: cubic centimeters per second per square centimeter. Then, when the values of η_0, ρ_0, and g_0, expressed in c.g.s. units, are supplied, the value of k is obtained directly.

For the conversion of P from Meinzer units to metric units

$$1 \frac{\text{gal/day}}{\text{ft}^2} = 4.716 \times 10^{-5} \frac{\text{cm}^3/\text{sec}}{\text{cm}^2} .$$

We may take the standard values of η_0 and ρ_0 (the viscosity and density of water at 60° F.) to be 0.01125 poise and 0.999 gm/cm³, respectively; and for a standard value of gravity we may ues 980 cm/sec². From these values we obtain

$$\frac{\eta_0}{\rho_0 g_0} = \frac{0.01125}{0.999 \times 980} \text{ cm-sec} = 1.149 \times 10^{-5} \text{ cm-sec} .$$

Then the value of k, expressed in square centimeters, is

$$k[\text{cm}^2] = (1.149 \times 10^{-5} \times 4.716 \times 10^{-5}) P \left[\frac{\text{gal/day}}{\text{ft}^2}\right]$$
$$= (5.419 \times 10^{-10}) P \left[\frac{\text{gal/day}}{\text{ft}^2}\right] .$$

Also, since 1 ft.^{2} = 929.0 cm.^{2}, the value of k expressed in square feet is given by

$$k[\text{ft}^2] = \left(\frac{5.419 \times 10^{-10}}{929.0}\right) P \left[\frac{\text{gal/day}}{\text{ft}^2}\right]$$

$$= (5.833 \times 10^{-13}) P \left[\frac{\text{gal/day}}{\text{ft}^2}\right].$$

Conversely,

$$P \left[\frac{\text{gal/day}}{\text{ft}^2}\right] = (1.845 \times 10^9) k[\text{cm}^2] = (1.714 \times 10^{12}) k\,[\text{ft}^2].$$

It is to be emphasized that P and k are two entirely different kinds of quantity and therefore cannot be equated. It is meaningless to ask: "How many units of k are *equal* to 1 unit of P?" The foregoing conversions do not therefore represent a relationship of equality but of function. The common bond between P and k is the permeability of the medium; for a given permeability the values of both P and k are uniquely determined. Consequently, if the value of either, expressed in any arbitrary set of units, is known, the value of the other, expressed in any other arbitrary set of units, can be determined.

The measured values of the permeability of sediments, expressed in Meinzer units, range from the order of 10^{-4} for clayey silt to 10^{+5} for gravel. The corresponding range of the permeability measured in square centimeters would be from the order of 10^{-14} to 10^{-5}, and measured in square feet from 10^{-17} to 10^{-8}. It is obvious, therefore, that a permeability of 1 square centimeter or of 1 square foot is very much larger than the permeabilities encountered in practice.

This, however, presents no serious difficulty. It has become almost the universal practice to express the results of physical measurements in the form of a product of a small decimal number with 10 raised to an integral power, positive values of the exponent corresponding to numbers greater than 1, and negative values to those less than 1.

It is equally easy in this manner to deal with numbers expressed as a product of 10 raised to either a large or a small or to a positive or a negative exponent.

Still, if the need for a smaller "practical" unit be felt, there are two ways of achieving such a unit. One of these is to change the fundamental units employed; instead of the centimeter, the gram, and the second, suitable multiples and submultiples of these might be used. This, however, is likely to be confusing, and it is far simpler to retain the fundamental units and define the "practical" unit to be

$$1\ [\text{practical unit}] = 10^n\ [\text{cm}^2] = 1[10^n\ \text{cm}^2]\ .$$

The choice of n is entirely arbitrary and is dependent upon the properties it is desired to impart to the practical unit. For example, if a practical unit is desired such that all measured values of permeability would be multiples of this unit, then n should have the value of about -15. If a unit is wanted such that the greatest number of measured values would range between 1 and 1,000, then n should be about -10.

THE "LAW" OF BADON GHIJBEN AND HERZBERG

Of late a great deal of emphasis has been placed upon the so-called "law" of Badon Ghijben[21] and Herzberg.[22] Each of these authors independently made the discovery that in wells near the seacoast the salt water was not encountered at sea-level, as they had expected, but at a depth below sea-level of the order of forty times the height of the fresh water above sea-level. For this phenomenon each of them deduced the same explanation, namely, that a static equilibrium existed between the fresh water and the salt water, so that the mass of a unit vertical column of fresh water extending from the water table to the fresh-water–salt-water interface must have the same mass as the displaced salt water—a column extending from the interface to sea-level. Following this reasoning, if we let z'_{12} be the elevation of the water table above sea-level, z'_{23} that of the fresh-water–salt-water interface directly below, ρ_2 the density of fresh water, and ρ_3 that of salt water, we should have

$$\rho_2(z'_{12} - z'_{23}) + \rho_3 z'_{23} = 0\ ,$$

[21] W. Badon Ghijben, "Nota in Verband met de Voorgenomen Put boring Nabij Amsterdam," *Tijdschrift van het Koninkhijk, Institut van Ingenieurs*, 1888–89, p. 21.

[22] Baurat Herzberg, "Die Wasserversorgung einiger Nordseebader," *Journal für Gasbeleuchtung und Wasserversorgung*, Vol. XLIV (1901), pp. 815–19 and 842–44.

from which

$$z'_{23} = -\frac{\rho_2}{\rho_3 - \rho_2} \cdot z'_{12} . \qquad (291)$$

Equation (291) is equivalent to those deduced by both Badon Ghijben and Herzberg. It is a relationship of great usefulness but also one whose use entails considerable danger of error, because it is not correct.

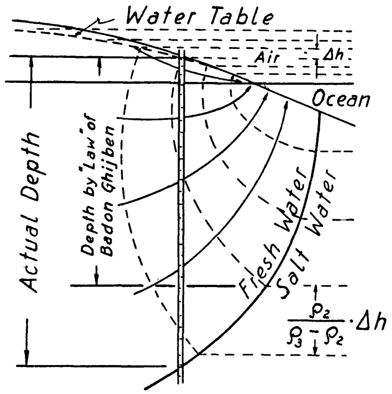

FIG. 42.—Discrepancy between actual depth to salt water and depth as calculated by "law" of Badon Ghijben and Herzberg for flow near outlet.

The assumption upon which the equation is derived is that we are dealing with a case of hydrostatic equilibrium, and to the extent that this assumption is valid the equation is strictly correct. In ground-water problems, however, the assumption is not valid at all, because the fresh water is not at constant potential but in a state of continuous motion. If no additional water were added by precipitation, the flow of the fresh water would continue until it had all been dissipated, and only salt water with a water table at sea-level would remain.

We are therefore dealing with a dynamic equilibrium between flowing fresh water and static salt water, the behavior of which has already been treated in equations (205)–(217). It will be noted that equations (291) and (215) are identical except that in (215) z_{12} is the elevation in air of a particular fresh-water equipotential surface, and z_{23} the elevation in the salt water of the same surface, whereas in equation (291) z'_{12} is the water-table elevation and z'_{23} the elevation of the fresh-water–salt-water interface in the same vertical line.

For static conditions equation (215) is identical in all respects with (291). For low potential gradients in the fresh water the difference between the two equations is negligible; but for large gradients, such as occur near a pumping well, a canal, or the seacoast, the static relationship of equation (291) may not be even approximately correct, and its use is not warranted under those conditions. In fact, errors can best be avoided by always using the dynamic equations of which a static situation is only a special case.

APPLICATION TO PROBLEMS OF CURRENT INTEREST

Before closing, it will be instructive to examine a few illustrative ground-water problems of current interest in the light of the present theory. The most useful problems for this purpose are those upon which disagreements have arisen or upon which opinions contrary to the present deductions have been held. Since the present theory is based upon the principle of the conservation of matter and the laws of thermodynamics, to which ground-water motion must conform, it is to be expected that a postulated motion of ground water not in conformity with present deductions can be demonstrated to violate one or another of these principles—a violation of either of the laws of thermodynamics being equivalent to a perpetual-motion mechanism.

FLOW NEAR THE WATER TABLE

One such problem is the nature of the flow near the water table. Apparently because of the resemblance between the topography of the water-table surface and that of the surface of the ground, it used to be commonly supposed, and the idea still prevails to some extent, that the ground-water flow is concentrated near the surface of the water table and that the flow down the slope of the water table resembles the corresponding flow down a topographic slope. The intensity of flow was supposed to diminish rapidly with depth, becoming substantially zero in a region of more or less uniform permeability at depths below the level of the lowest parts of the water table. This led to the conception of a large body of stagnant ground water beneath the superficial zone of flow.

This point of view has recently been defended by Swinnerton.[23] Taking a region of mild topographic relief in a humid climate, with rocks of uniform permeability, Swinnerton argues that from a point *A* (Fig. 43) (shown as a volume element beneath the water table) the water will flow, by all possible paths, to an outlet (shown as a near-by valley). The flow will be distributed among those several paths according to the frictional resistance, the greatest amount of flow following the path of least resistance, and successively lesser amounts the paths of increasing resistance. The path of least resistance is the shortest path, and the resistance increases with increase of length of path. Among the paths 1, 2, and 3 of increasingly greater length, the flow will be greatest along the rectilinear path 1 and successively less along the paths 2 and 3.

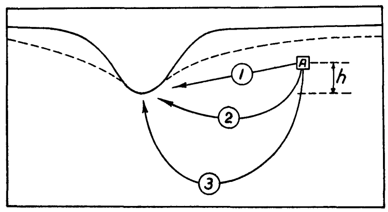

FIG. 43.—Flow from region *A* to outlet according to Swinnerton.

Aside from any question of frictional resistance, this involves directly a violation of the principle of the conservation of matter. If flow occurs from *A* by all possible paths, of which 1, 2, and 3 are examples, then *A* will be a volume out of which water flows in all directions and hence a region in which water must be created.

If this postulated flow mechanism were true of an arbitrary point *A*, then it must also be true of a near-by point *A′*, and the families of flowlines radiating from these two points must form intersecting systems throughout the near-by field. Since two flowlines can only come together at an absolute source or sink (point where water is created or annihilated), which does not exist in ground-water motion, this type of flow would involve the violation of the conservation of matter principle at all points of the field.

Again, no amount of frictional resistance can be assigned to any stream tube because the resistance is a function both of the geometry of the tube and the *rate of flow of the fluid.* We know, for example, that the amount of energy expended per unit of mass by frictional

[23] A. C. Swinnerton, "Origin of Limestone Caverns," *Geol. Soc. Amer. Bull. 43* (1932), pp. 663–92.

forces is precisely equal to the drop in potential. Therefore, along all streamlines, of whatever shape or length, the amount of work required to drive a unit of mass of the fluid from one equipotential surface to another is precisely the same, although the paths followed by different flowlines may differ widely as to length.

We can get at what actually happens in a case of this sort only by giving due consideration to the distribution of the sources and sinks upon the boundaries of the flow field, and of the corresponding potentials. Here, as explained earlier, a source merely means an area across which water enters the field; and a sink, an area of exit.

In this problem the sources are distributed over the air-water interface, being produced by downward percolating water. The sinks are limited to the bottoms of valleys containing streams. In the field of flow the streamlines and stream tubes originate upon sources and terminate upon sinks. The flow is solenoidal and the velocity finite, so that every stream tube of finite discharge has a finite cross section, and no two flowlines can intersect each other. At a given time one and only one flowline can pass through a given point. All of these are deductions from the principle of the conservation of matter.

Now for the energy relations. The potentials at the sources and sinks are everywhere given by $\Phi = gz$, where z is the elevation of the water table or of the surface of the water in a stream. In air, the surfaces $\Phi =$ constant coincide with surfaces $z =$ constant; underground they refract and are everywhere normal to the streamlines. At the sinks the cross section of a stream is sensibly an equipotential region, so that underground the equipotential surfaces about a sink must form a system of more or less concentric surfaces with a radiating system of flowlines in the vertical plane normal to the valley axis.

Since every stream tube has its source on the air-water interface, it is necessary that it intersect this interface at some angle greater than zero, for otherwise its discharge would be zero. We can form some idea of the angle of slope of the stream tube at the air-water interface if we take into account that above this interface there is a vertically downward percolation at a specific discharge of q_1 under the gradient of $-g$. The fact that the fluid in this region is discontinuous shows that the discharge is less than that demanded by Darcy's law. Hence,

$$q_1 < -\sigma g \tag{292}$$

and the component normal to the interface is given by

$$q_{n1} < -\sigma g \cos \alpha \,, \tag{293}$$

where α is the upward slope of the air-water interface.

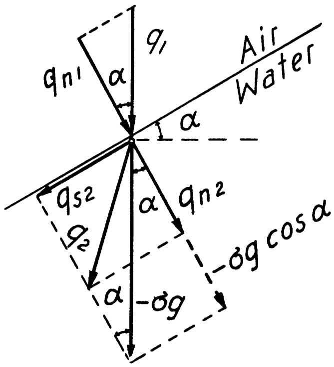

FIG. 44.—Flow vectors at air-water interface when downward infiltration of water occurs.

Along the interface the usual boundary conditions must prevail. The normal component of the flow and the tangential component of the potential gradient must be the same. Thus,

$$q_{n1} = q_{n2} < -\sigma g \cos \alpha \,, \tag{294}$$

$$\left(\frac{\partial \Phi}{\partial s}\right)_1 = \left(\frac{\partial \Phi}{\partial s}\right)_2 = g \sin \alpha \,. \tag{295}$$

Consequently the tangential component of flow on the water side of the air-water interface is

$$q_{s2} = -\sigma g \sin \alpha \,. \tag{296}$$

Equations (294) and (296) give the tangential and normal components of the flow at the air-water interface. The vector sum of these is $\mathbf{q}_2$. The normal component q_{n2} may vary between the limits zero and $-\sigma g \cos \alpha$, and α between the limits of zero and 90°. As $\mathbf{q}_{n2}$ approaches zero, $\mathbf{q}_2$ approaches $\mathbf{q}_{s2}$ as a limit, and the flowlines become tangential to the interface. As q_{n2} approaches its upper limit, $-\sigma g \cos \alpha$, $\mathbf{q}_2$ approaches the limit $-\sigma \mathbf{g}$, with the flowlines vertical. For a constant value of $\mathbf{q}_{n2} \neq 0$, as α tends to zero q_{s2} tends to zero, and $\mathbf{q}_2$ tends to $\mathbf{q}_{n2}$, which again is vertical.

Hence, in Swinnerton's problem the flowlines at their points of origin on the air-water interface always dip more steeply than the interface itself, becoming vertical at the ground-water divide, where

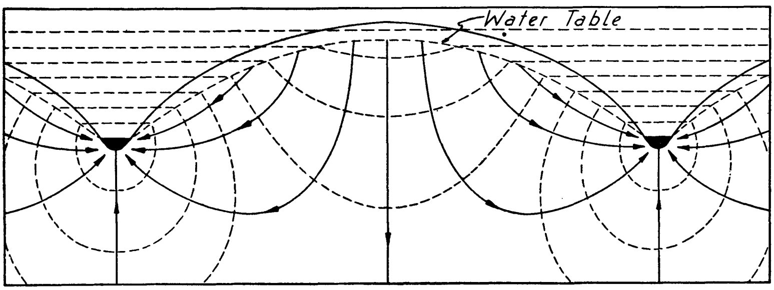

FIG. 45.—Approximate flow pattern in uniformly permeable material between the sources distributed over the air-water interface and the valley sinks.

α is zero and, across the divide, dipping in the opposite direction. Accordingly, at the divide the stream tubes descend vertically; and away from the divide, toward the valleys on either side, the angles of pitch of the flowlines gradually approach the angle of dip of the interface. The stream tubes adjacent to the outlets follow almost rectilinear paths, while each tube originating successively farther away makes a downward loop outside the paths of all stream tubes between it and the outlet, until finally the one at the divide descends to a depth which in uniformly permeable material has no assignable limit.

In such a region there can exist no body of stagnant water unless it be a body of water such as salt water, which is physically different from that flowing. The region of most intense flow is not a zone parallel to the air-water interface but a region where the flowlines converge toward a sink.

THE JOHNSON HYPOTHESIS OF THE ORIGIN OF SUBMARINE CANYONS

For a problem involving energy relationships more directly we turn now to the hypothesis advanced by Johnson[24] of the origin of the submarine canyons off the east coast of North America. All that interests us here is that this hypothesis involves a ground-water

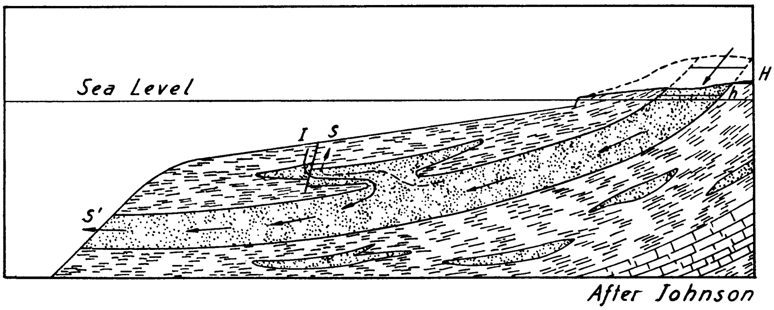

FIG. 46.—Johnson's Fig. 2

mechanism shown in Figure 46, taken from Johnson and described by him (p. 80) as follows:

> One may suppose, for example, that artesian water with a head H (Figure 2) escapes in part (dotted arrows) across adjacent pervious beds and through fissures or other openings to form a shallow submarine spring at S; and in part as deep submarine springs at S'. When erosion reduces the land surface, or subsidence lowers it, or rise of sea level occurs, the lesser head h may be insufficient to cause outflow. Under the new conditions, salt water flowing in through the fissure, now serving as an intake, I, will mix with fresh water in the aquifer, and the mingled waters will find exit at S'. It should be borne in mind that while fresh water descending along a fissure or through pervious beds will not intermingle freely with heavier salt water below, but rather will "float" upon it, salt water penetrating downward in the same manner, tends, because of its higher specific gravity, to enter fresh water below and spread through it, increasing the weight of the mixture.

Without any deeper analysis, the first thing that impresses one about this mechanism is that, if the denser salt water is to penetrate downward and intermingle with the fresh water, and then this mixture, which is still less dense than salt water, is to continue its

[24] Douglas Johnson, *The Origin of Submarine Canyons* (New York: Columbia University Press, 1939).

descent to an outlet at S', we should also expect the denser cold air at the top of a chimney to penetrate downward and intermingle with the warm air in the chimney, and the mixture to continue downward and find an exit through the fireplace.

Now to handle the problem analytically, let ρ_1, ρ_2, and ρ_3 be the densities of the fresh water, the mixture, and the salt water respectively, and Φ_1, Φ_2, and Φ_3, the corresponding potentials. These potentials at all points are

$$\left.\begin{aligned} \Phi_1 &= gz + \frac{p - p_0}{\rho_1}\,, \\ \Phi_2 &= gz + \frac{p - p_0}{\rho_2}\,, \\ \Phi_3 &= gz + \frac{p - p_0}{\rho_3}\,; \end{aligned}\right\} \tag{297}$$

and at the points of contact with the salt water,

$$p = -\rho_3 gz + p_0\,, \tag{298}$$

which gives as the values of the potentials at those points:

$$\left.\begin{aligned} (\Phi_1)_3 &= -\frac{\rho_3 - \rho_1}{\rho_1}\cdot gz\,, \\ (\Phi_2)_3 &= -\frac{\rho_3 - \rho_2}{\rho_2}\cdot gz\,, \\ (\Phi_3)_3 &= \frac{\rho_3 - \rho_3}{\rho_3}\cdot gz = 0\,. \end{aligned}\right\} \tag{299}$$

From this it is clear that both $(\Phi_1)_3$ and $(\Phi_2)_3$ increase with the depth below sea-level.

Now let P_1 be the point on the main aquifer where the two channels separate, P_2 a point at the zone of mixing, and P_3 the lower outlet (Johnson's S'). By Johnson's hypothesis

$$\Phi_{13} > \Phi_{11} > \Phi_{12}\,, \tag{300}$$

so that the fresh water of itself is not able to flow out at P_3. At P_2, however, mixing occurs, and the mixture then flows from P_2 to the outlet at P_3.

To determine whether this is possible, we only have to observe that the point P_2, where the mixing occurs, must be somewhere higher than P_3. At these two points the potentials of the mixture are

$$\left.\begin{aligned} \Phi_{22} &= -\frac{\rho_3 - \rho_2}{\rho_2} \cdot g z_2 \,, \\ \Phi_{23} &= -\frac{\rho_3 - \rho_2}{\rho_2} \cdot g z_3 \,, \end{aligned}\right\} \tag{301}$$

whose difference is

$$\Phi_{23} - \Phi_{22} = -\frac{\rho_3 - \rho_2}{\rho_2} \cdot g(z_3 - z_2) > 0 \,. \tag{302}$$

Hence the flow from P_2 to P_3 is from a lower to a higher potential.

We are obliged to conclude, therefore, that the flow mechanism postulated by Johnson, if it could be made to work, would violate the principle of the conservation of energy or the first law of thermodynamics, and hence constitutes a perpetual-motion mechanism of the first kind. In principle it is no different from water flowing slowly uphill in an open channel.

What actually would happen in the situation shown by Johnson's Figure 2 would be this: The fresh-water–salt-water interface would occur at an elevation

$$z_{13} = -\frac{\rho_1}{\rho_3 - \rho_1} \cdot h \,,$$

obtained from equation (212), where h is the fresh-water manometer height above sea-level at the point of contact. This h is equal to, or less than, the height of the water table used by Johnson, depending upon whether the water is static or in motion. If

$$z_{13} > z_1 \,,$$

the interface will be higher than P_1, and no flow will occur. If

$$z_3 < z_{13} < z_1 \,,$$

fresh water will flow out at S and a static interface will exist between P_1 and P_3. If

$$z_{13} < z_3 \,,$$

fresh water will flow out at both S and S'.

THE FLORIDA SHIP CANAL

A somewhat more complex problem is afforded by the proposed Florida ship canal, pertaining to which a summary of the essential data has been presented by Paige[25] of the United States Army Corps of Engineers. The proposed canal is to be unlined and at sea-level, and a part of the route chosen extends across the outcrop area of the Ocala limestone. This is a highly permeable limestone and is the principal aquifer of Florida. It outcrops over an area which is roughly a north-south ellipse about 150 miles long and 50 miles wide, and forms the central region of a huge structural dome away from which the Ocala dips and underlies the younger formations of the rest of the state.

About 50 miles to the north of the canal route and 78 miles to the south there occur two large highs in the piezometric surface based upon the Ocala limestone, the one to the north reaching over 90 feet and the one to the south about 135 feet above sea-level. In the outcrop area the piezometric surface merges into the ground-water table. The piezometric highs coincide with topographic highs and are only slightly lower than the water-table surface. In the outcrop area the canal route extends across the saddle of the ground-water table between these two highs. In this region the elevation of the water table is about 40 feet above sea-level.

The principal ground-water problem involved, and the only one we shall consider here, is: What effect will the canal have upon the ground-water table in its vicinity where it traverses the outcrop of the Ocala limestone, and how wide will the belt on both sides of the canal be in which the effect is perceptible in wells? Paige concluded that a new ground-water table with a surface steepest near the canal and approaching the original water table asymptotically with distance would result. The distance from the canal to where this new water table would coincide with the older one Paige estimated to be about 10 or 15 miles.

In this conclusion he has been severely criticized by Brown[26] and

[25] Sidney Paige, "Effect of a Sea-Level Canal on the Gound-Water of Florida," *Econ. Geol.*, Vol. XXXI (1936), pp. 537–70.

[26] John S. Brown, "The Florida Ship Canal," *Econ. Geol.*, Vol. XXXII (1937), pp. 589–99.

by Thompson, Meinzer, and Stringfield,[27] to whom Paige[28] later replied.

Since this is a problem upon which there appears to be substantial agreement with regard to the principal field data, the disagreements evidently arise from a difference in the theory employed in making the predictions. It will therefore be instructive to examine it in the light of the theory developed here.

The only problem we shall consider is that of the new profile of the water table where the canal traverses the Ocala limestone, and the maximum distance to which the lowering of the water table will be appreciable. Only an approximate solution is possible with available data, but one of the correct order of magnitude should be obtainable.

For our approximation we will idealize the problem somewhat by considering a case where the topography and water table are the same as those of Florida but where the rocks are all of Ocala limestone and of uniform specific conductivity σ. We let h' be the original elevation of the water table and h that of the new water table after the canal is dug. The value of h' at the canal route in the saddle is about 40 feet and at the southern high, 78 miles to the south, slightly more than 135 feet, giving an average water-table slope of about 1.2 feet per mile.

Since this is the smallest slope and the highest water table to be encountered, the effect of the canal both in terms of the amount the water table is lowered and the width of the zone affected should here be a maximum. Since we are interested chiefly in this maximum value, we need not consider other cases.

The canal, after it is dug, will act as a horizontal prismatic sink containing water at constant potential with a free surface only slightly above sea-level. The fresh-water flowlines will converge toward this prism perpendicularly to its surface. They will therefore be parallel to a plane normal to the prismatic axis and will approach the canal more or less radially in this plane.

[27] D. G. Thompson, E. O. Meinzer, and V. T. Stringfield, "Effect of a Sea-Level Canal on the Gound-Water Level of Florida," *Econ. Geol.*, Vol. XXXIII (1938), pp. 87–107.

[28] Sidney Paige, "Effect of Sea-Level Canal on the Gound-Water Level of Florida—A Reply," *Econ. Geol.*, Vol. XXXIII (1938), pp. 647–65.

The air–fresh-water interface will slope toward the canal and approach the south bank tangentially (presumably at about 30°). There will be a surface of seepage between this and the water-level in the canal of a finite but unknown width. The fresh-water–salt-water interface will slope upward toward the canal; and, since the flow will be almost bilaterally symmetrical on opposite sides of the canal, the salt water will form a sharp wedge, with its crest parallel to the axis of the canal and reaching to within a few tens of feet of its bottom.

The magnitude of the slopes of both the upper and lower interfaces will decrease rapidly with distance away from the canal; and at some unknown distance, which we shall seek to estimate, the new water table will approach coincidence with that which existed before.

To determine the properties of this new profile analytically, we choose the origin of co-ordinates at sea-level on the axis of the canal and let the x-axis extend southward normal to the canal and the z-axis upward. We take as our flow region a laminar section parallel to the x–z plane and of unit thickness. At some distance, x_1, which we shall seek to determine, the new water-table profile will come into coincidence with the old one. Between this distance and the canal we assume that there are no sources or sinks, so that over this interval the total discharge, Q, is constant. We let h be the elevation of the new water table and A the cross-sectional area of a stream tube of unit thickness normal to the flowlines at any distance x. Equating the discharge at x and x_1 then gives

$$Q = qA = -\sigma g \cdot \frac{dh}{dx} \cdot A = -\sigma g \left(\frac{dh}{dx}\right)_1 A_1 . \qquad (303)$$

Eliminating $-\sigma g$ and solving for dh/dx gives

$$\frac{dh}{dx} = \left(\frac{dh}{dx}\right)_1 A_1 \cdot \frac{1}{A} . \qquad (304)$$

Setting $(dh/dx)_1 A_1 \equiv a$ and integrating,

$$h = h_0 + a \int \frac{1}{A} \cdot dx . \qquad (305)$$

In order to perform this integration and determine h we must find how the area A varies with distance along the profile. In the immediate vicinity of the canal this is not easy to do though we know that the smallest value of A is equal to one-half the wetted perimeter of the canal. But for the part of the profile where the ground-water slope is not more than 100 feet per mile, we may assume, as an approximation, that the area A is equal to the difference of the elevations in air and salt water of the fresh-water equipotential surface occurring at distance x. This is obtained from equation (213) and

$$A \cong (z_{12} - z_{23}) = \frac{\rho_3}{\rho_3 - \rho_2} \cdot h = bh \,, \qquad (306)$$

where $b \equiv \rho_3/(\rho_3 - \rho_2)$.

Let x_0 be the distance at which this approximation becomes allowable, and h_0 the height of the water table at that distance. Then, if we introduce the value for A from equation (306) into equation (304), the elevation of the profile between x_0 and x_1 is obtainable by integration. The differential equation of the profile is

$$\frac{dh}{dx} = \frac{a}{b} \cdot \frac{1}{h} ,$$

and separating the variables and integrating,

$$\int_{h_0}^{h} h \cdot dh = \frac{a}{b} \int_{x_0}^{x} dx = \frac{h^2 - h_0^2}{2} = \frac{a}{b} (x - x_0) \,,$$

from which

$$h^2 = \frac{2a}{b} (x - x_0) + h_0^2 \,. \qquad (307)$$

Since all factors of this equation except h^2 and x are constants, this represents a straight line with the slope $2a/b$, passing through the point (x_0, h_0^2), when plotted as a graph using the values of h^2 as ordinates and x as abscissas.

If on the same graph we plot the values of h'^2 of the old water-table profile, the line represented by equation (307) must be a tangent line to this, the abscissa of the point of tangency being x_1. Through any given point (x_0, h_0^2) only one tangent line to the old profile can be drawn, and by means of this x_1 and a are uniquely determined. The value of b is given by the densities of fresh and salt water and is about 40.

We can only approximate x_0 and h_0. Since the area A decreases continuously as the canal is approached, the slope of the water table, as given by equation (304), must continuously increase, that is, at the canal the slope has its maximum value (30° or so) and decreases continuously with distance. Hence, the average slope from the canal to any distance x is always greater than that at x. Also, at any distance x,

$$0 < h < h' ;$$

and at x_0,

$$0 < h_0 < h'_0 .$$

We chose x_0 as the distance at which the water-table slope becomes 100 feet per mile. Between $x = 0$ and $x = x_0$ the average slope must be greater than 100 feet per mile. At the same time h_0 must be less than the elevation of the old water table, or less than 40 feet. Therefore x_0 must be less than $\frac{1}{2}$ mile from the south bank of the canal.

To be conservative, let x_0 be taken to be 1 mile. At this distance h_0 will be greater than zero by an unknown amount, possibly by 10 or 20 feet; but again, in order to be conservative, let us set it equal to zero. Then the squared profile of the new water table is the straight line passing through the point (1, 0) and tangent to the h'^2 profile of the old water table.

These operations have been performed and the results plotted graphically in Figure 47. Coincidence between these profiles occurs at about 23 miles south of the canal at an elevation of about 60 feet. At 20 miles the lowering of the water table is negligible; at 15 miles the drawdown is 2 feet (from 52 to 50 feet); at 10 miles, 5 feet (from 45 to 40 feet); and at 5 miles, 15.5 feet (from 42 to 26.5 feet).

Now is the time to examine our assumptions. We assumed that the Ocala limestone was of uniform permeability and unlimited thickness. Locally, of course, owing to solution channels or other inhomogeneities, the permeability is not uniform; yet the smoothness of the hydrologic contours on the Ocala shows that regionally it does have an effective permeability uninfluenced by such local irregularities. It is the effective regional permeability that we have employed here, and we have so arranged our equations that it cancels out and saves us the necessity of knowing its value.

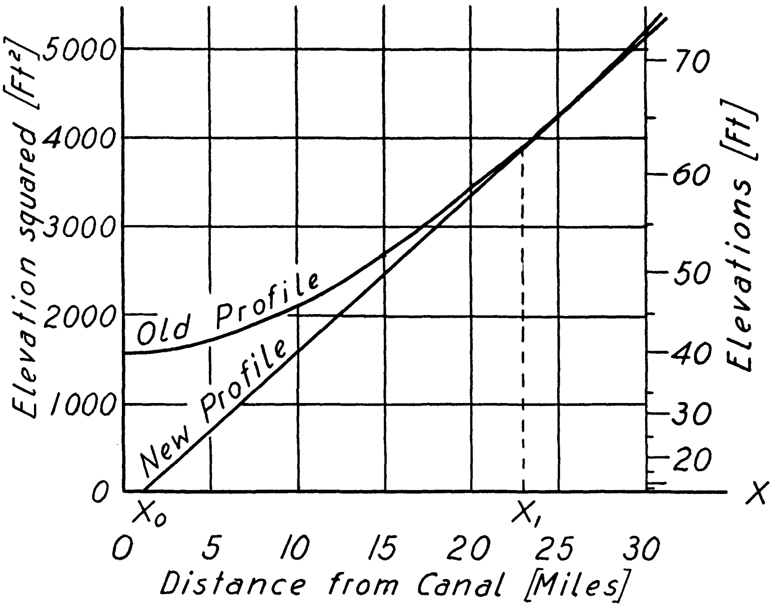

FIG. 47.—Comparison of the original ground-water profile south of the Florida ship canal with the limiting value of the new one as calculated here.

With the assumption that the Ocala limestone is of unlimited thickness, the area A continuously increases with distance from the canal because of the lowering of the salt-water interface. At 23 miles this represents a depth of about 2,400 feet. If the Ocala is less than this thickness and is underlain by impermeable rocks, then x_1 will very nearly coincide with the distance at which the salt-water interface reaches the lower surface of the limestone, and the width of the affected zone will then be narrower than that calculated here.

We assumed that no sources and sinks existed from the canal to the distance x_1 and that Q was constant over this length. This assumption is false, but again the error is on the conservative side. Owing to precipitation, there will be distributed sources all along the profile. There will also be sinks where the water discharges locally into springs and streams. These, however, will diminish as the canal is approached and the water table is lowered, so that, on the whole, the difference between sources and sinks, or the net sources, will increase toward the canal. The effect of this will be to cause Q to increase toward the canal, making the water-table

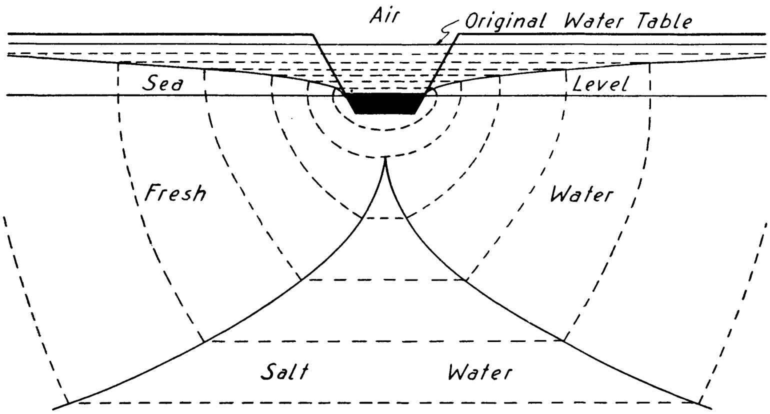

FIG. 48.—Schematic diagram showing approximately the type of relationship between the bodies of air, fresh water, and salt water to be expected near a sea-level canal in uniformly permeable material. The scales are greatly distorted. The depths to salt water should be four times those shown here.

profile higher at all points than that which we have computed. This will correspondingly reduce the amount of drawdown at each distance. Also, the curve of h^2 plotted as a function of x will no longer be a straight line but will be an arc convex upward. The point of tangency with the old profile will therefore also be nearer to the canal than that which we have computed.

We assumed that at a distance of 1 mile from the axis of the canal the elevation of the new water-table profile was zero. Since, at the south bank of the canal, the profile has an initial elevation equal to that of the upper edge of the zone of seepage and an initial slope equal to that of the south bank (30° or more), and since the slope remains positive, decreasing at an unknown rate with the distance, then its elevation at a mile from the axis of the canal must be greater than zero by some unknown amount—possibly 10 or 20 feet. It is clear, therefore, that the assumption that at this distance the elevation of the profile is zero is in error, and perhaps widely so. Again, however, the error is on the conservative side, and all points on the actual profile must be higher than those given on the basis of this assumption, and the true distance of approximate coincidence correspondingly nearer.

Our assumptions have therefore all been made on the conservative side, so that the elevations of the profile shown in Figure 47 are lower and the distance x_1 is greater than would be the case for the actual profile. The present values are, therefore, extreme ones and are intended to establish a limit to the effects to be expected.

A drawdown less than 5 feet at 10 miles and 2 feet at 15 miles is of negligible importance in its effect upon local wells and water supply, because at these distances the depth to salt water would still be 1,600–2,000 feet below sea-level.

Hence, the estimate made by Paige and his colleagues that the zone of appreciable lowering of the water table would not extend farther than about 10 or 15 miles from the canal is entirely confirmed by the present analysis and appears to be a conservative estimate. The misgivings of Paige's critics, one of whom (Brown) predicted that water-table slopes of 1–2 feet per mile with a water table only slightly above sea-level over a zone 20–40 miles wide would result, appear, so far as this problem is concerned, to be entirely unfounded.

CONCLUSION

While the analytical theory of ground-water motion has made great progress during the last fifty years toward becoming a branch of exact physical science, we have found that progress has been continuously impeded by certain persistent fundamental misconceptions regarding the potential function governing ground-water motion. It has been our attempt here to re-examine the fundamentals of this subject from first principles and to establish it upon a basis more in conformity with the physical principles governing the motion of terrestrial matter—the principle of the conservation of matter and the laws of thermodynamics—than has been the case with the earlier treatments.

In the present paper the effort has been directed primarily at deriving fundamental relationships of interest and importance when dealing with problems of ground-water hydrology. In doing this, advantage has been taken of the close analogy between many aspects of ground-water theory and electrical theory, so that the relationships established are, as nearly as possible, expressed in a form already familiar to students of electricity. Thus Darcy's law, as here established, is strictly analogous to Ohm's law in electricity. Fluid potential is analogous to electrical potential; the flow vector $\mathbf{q}$ is analogous to the electrical current vector $\mathbf{i}$; the specific fluid conductivity is analogous to specific electrical conductivity; and the tangent law of refraction is common to both fluid and electrical flow. By means of these analogies treatises of electricity become among our most authoritative texts upon ground-water motion and are commended as such to students of this subject. On the other hand, multifluid problems have no counterpart in electricity and so have been given special attention here.

In the development of the present theory every effort has been made to satisfy a twofold objective: to so express the results obtained that they may easily be employed for the attainment of quick, graphical, approximate solutions of ground-water problems; and at the same time to formulate the basic equations in such a manner as to make them readily adaptable to the more precise analytical treatments of two- and three-dimensional scaler and vector fields.

For most ground-water problems rapid graphical solutions are possible which will give results of the desired accuracy. These involve the drawing of the flow fields in one or more principal vertical cross sections in conformity with the geometry and permeability of the medium, and the known boundary conditions. For this purpose, rigid observance of a small number of fundamental rules will prevent the making of serious blunders. Among the more important restrictions to be observed are these:

1. At points where the velocity is not zero, one and only one flowline can pass through the same point at the same time.

2. Ground-water flow is solenoidal, and the discharge along any stream tube is constant.

3. A stream tube cannot converge to zero and can only terminate upon permeable boundaries of a field of flow or else within the field by having the velocity approach zero as the cross-sectional area of the tube becomes unlimitedly large.

4. The normal component of the flow vector **q** on both sides of all surfaces is the same.

5. The flow vector **q** at every point in an isotropic permeable medium is given in both direction and magnitude by

$$\mathbf{q} = -\sigma \, grad \, \Phi .$$

The flowlines are everywhere normal to the equipotential surfaces, and the intensity of the flow increases as the distance between equidifferent equipotential surfaces decreases.

6. No two equipotential surfaces of different potentials can intersect.

7. No equipotential surface can close completely upon itself.

8. No equipotential surface can terminate except upon a boundary of the field of flow.

9. The tangential components of the potential gradient on opposite sides of any permeable surface are the same.

10. When ground water flows across a plane interface between rocks of different permeabilities, the flowlines and equipotential surfaces refract by the tangent law.

11. A static or slowly moving body of water occupying a void or basin is at constant potential.

12. The flowlines terminate perpendicularly upon the permeable surfaces of all equipotential regions.

13. Equipotential surfaces terminate perpendicularly upon all impermeable boundaries of a field of flow.

14. When two fields of flow are superposed, the resultant flow vector at each point is the vector sum of the vectors of the component fields, and the resultant potential field is the algebraic sum of the component potentials.

15. When fresh water coexists with bodies of static air and salt

water, if the fresh water is static the interfaces are horizontal, and the sequence of the fluids in the upward direction is in the order of decreasing density.

16. If the fresh water flows, the interfaces will both tilt toward the fresh-water layer in such a manner as to decrease its area of cross section in the direction of the flow.

17. Equidifferent fresh-water equipotential surfaces refract at the interfaces with static bodies of air and salt water into equally spaced horizontal surfaces, the spacing in salt water being $\rho_2/(\rho_3 - \rho_2)$ times the spacing in air, where ρ_3 and ρ_2 are the densities of salt and fresh water, respectively.

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